

STACKS OF FIBER FUNCTORS AND TANNAKA'S RECONSTRUCTION

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ABSTRACT. Given a quasi-compact category fibered in groupoids \mathcal{X} and a monoidal subcategory \mathcal{C} of its category of locally free sheaves $\text{Vect}(\mathcal{X})$, we are going to introduce the stack of fiber functors $\text{Fib}_{\mathcal{X},\mathcal{C}}$ with source \mathcal{C} , which comes equipped with a map $\mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \text{Fib}_{\mathcal{X},\mathcal{C}}$ and a functor $\mathcal{G}: \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}})$.

If \mathcal{C} generates $\text{QCoh}(\mathcal{X})$ and \mathcal{X} is an fpqc stack with quasi-affine diagonal, we show that $\mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \text{Fib}_{\mathcal{X},\mathcal{C}}$ is an equivalence, as it happens by Tannaka's reconstruction when \mathcal{X} is an affine gerbe over a field. In general, under mild assumption on \mathcal{C} , e.g. $\mathcal{C} = \text{Vect}(\mathcal{X})$, we show that $\text{Fib}_{\mathcal{X},\mathcal{C}}$ is a quasi-compact fpqc stack with affine diagonal and that the image $\mathcal{G}(\mathcal{C})$ generates $\text{QCoh}(\text{Fib}_{\mathcal{X},\mathcal{C}})$.

INTRODUCTION

Classical (non-neutral) Tannaka's duality over a field k establishes a correspondence between k -Tannakian categories and affine gerbes over k . More precisely given a k -Tannakian category \mathcal{C} we can define a fibered category $\Pi_{\mathcal{C}}: \text{Aff}/k \rightarrow (\text{groupoids})$ by

$$\Pi_{\mathcal{C}}(A) = \{k\text{-linear, strong monoidal and exact functors } \mathcal{C} \rightarrow \text{Vect}(A)\}$$

where Aff/k is the category of affine schemes over k and $\text{Vect}(A) = \text{Vect}(\text{Spec } A)$ is the category of finitely presented locally free sheaves on $\text{Spec } A$. It is then proved that $\Pi_{\mathcal{C}}$ is an affine gerbe and the functor

$$\mathcal{C} \rightarrow \text{Vect}(\Pi_{\mathcal{C}}), X \mapsto (\Gamma \ni \Pi_{\mathcal{C}}(A) \mapsto \Gamma_X \in \text{Vect}(A))$$

is a k -linear and strong monoidal equivalence. Here and in what follows we think sheaves (e.g. locally free, quasi-coherent) on a fibered category as functors from the fibered category itself to the corresponding fibered category of sheaves (see [Ton20, Section 1] for details).

Conversely if Δ is an affine gerbe over k then $\text{Vect}(\Delta)$ is a k -Tannakian category and the functor

$$\Delta \rightarrow \Pi_{\text{Vect}(\Delta)}, (s: \text{Spec } A \rightarrow \Delta) \mapsto (s^*_{|\text{Vect}(\Delta)}: \text{Vect}(\Delta) \rightarrow \text{Vect}(A))$$

is an equivalence. This is called Tannaka's reconstruction while the previous equivalence is called Tannaka's recognition.

In this paper we want to generalize Tannaka's reconstruction in two ways: consider more general fibered categories and proper subcategories of the category of locally free sheaves and introduce generalized stacks of fiber functors.

Fix a base commutative ring R . Consider a category fibered in groupoids \mathcal{X} over Aff/R and a monoidal subcategory $\mathcal{C} \subseteq \text{Vect}(\mathcal{X})$ that we assume is closed under taking duals. Let \mathcal{Y} be another category fibered in groupoids over R . Given an R -linear functor $\Gamma: \mathcal{C} \rightarrow \text{Vect}(\mathcal{Y})$ we say that Γ is right exact if it is exact on right exact sequences (in the ambient category $\text{QCoh}(\mathcal{X})$), whose terms are direct sum of sheaves in \mathcal{C} (see 3.1 for a precise definition). We define $\text{Fib}_{\mathcal{X},\mathcal{C}}(\mathcal{Y})$ as the groupoid of R -linear, right exact and strong monoidal functors $\Gamma: \mathcal{C} \rightarrow \text{Vect}(\mathcal{Y})$. We define the category fibered in groupoids $\text{Fib}_{\mathcal{X},\mathcal{C}}: \text{Aff}/R \rightarrow (\text{groupoids})$ by

$$\text{Fib}_{\mathcal{X},\mathcal{C}}(A) = \text{Fib}_{\mathcal{X},\mathcal{C}}(\text{Spec } A) = \{R\text{-linear, strong monoidal and right exact functors } \mathcal{C} \rightarrow \text{Vect}(A)\}$$

We will show that under mild hypothesis on \mathcal{C} the category $\mathrm{Fib}_{\mathcal{X},\mathcal{C}}$ is automatically fibered in groupoids even without forcing it (see 5.1). Notice also that $\mathrm{Fib}_{\mathcal{X},\mathcal{C}}$ is a stack for the fpqc topology. The analogy with classical Tannaka's duality is that $\mathrm{H}_{\mathrm{Vect}(\Delta)} = \mathrm{Fib}_{\Delta, \mathrm{Vect}(\Delta)}$ for an affine gerbe Δ over $R = k$ (see 3.4).

There are two natural functors. An R -linear, strong monoidal and right exact functor

$$\mathcal{G}: \mathcal{C} \rightarrow \mathrm{Vect}(\mathrm{Fib}_{\mathcal{X},\mathcal{C}}), \quad \mathcal{G}_{\mathcal{E}}(\Gamma \ni \mathrm{Fib}_{\mathcal{X},\mathcal{C}}(A)) = \Gamma_{\mathcal{E}} \in \mathrm{Vect}(A)$$

and a map of fibered categories

$$\mathcal{P}_{\mathcal{C}}: \mathcal{X} \longrightarrow \mathrm{Fib}_{\mathcal{X},\mathcal{C}}, \quad (s: \mathrm{Spec} A \rightarrow \mathcal{X}) \longmapsto (s|_{\mathcal{C}}^*: \mathcal{C} \longrightarrow \mathrm{Vect} A)$$

which generalize the maps defined in the case of affine gerbes.

We address two problems: when $\mathcal{P}_{\mathcal{C}}$ is an equivalence and, if this is not the case, what can we say about the category $\mathrm{Fib}_{\mathcal{X},\mathcal{C}}$. Notice that the composition $\mathcal{P}_{\mathcal{C}}^* \circ \mathcal{G}: \mathcal{C} \rightarrow \mathrm{Vect}(\mathcal{X})$ is the natural inclusion, so that, if $\mathcal{P}_{\mathcal{C}}$ is an equivalence, then \mathcal{G} is fully faithful.

By a quasi-compact fibered category we mean a category fibered in groupoids \mathcal{X} admitting a map $U \rightarrow \mathcal{X}$ from an affine scheme which is representable by fpqc covering of algebraic spaces. We say that a full subcategory $\mathcal{D} \subseteq \mathrm{QCoh}(\mathcal{X})$ generates $\mathrm{QCoh}(\mathcal{X})$ if any quasi-coherent sheaf on \mathcal{X} is a quotient of a direct sum of sheaves in \mathcal{D} . We prove the following generalization of classical Tannaka's reconstruction (see 3.4).

Theorem A (3.2, 3.3). *Let \mathcal{X} be a quasi-compact stack over R for the fpqc topology with quasi-affine diagonal and $\mathcal{C} \subseteq \mathrm{Vect} \mathcal{X}$ be a full, monoidal subcategory with duals generating $\mathrm{QCoh} \mathcal{X}$. Then $\mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \mathrm{Fib}_{\mathcal{X},\mathcal{C}}$ is an equivalence and, if \mathcal{Y} is a category fibered in groupoids over R , the functor*

$$\mathrm{Hom}(\mathcal{Y}, \mathcal{X}) \longrightarrow \mathrm{Fib}_{\mathcal{X},\mathcal{C}}(\mathcal{Y}), \quad (\mathcal{Y} \xrightarrow{f} \mathcal{X}) \longmapsto f|_{\mathcal{C}}^*: \mathcal{C} \longrightarrow \mathrm{Vect}(\mathcal{Y})$$

is an equivalence of categories.

In the case $\mathcal{C} = \mathrm{Vect} \mathcal{X}$, the functor $\mathcal{P}_{\mathrm{Vect}(\mathcal{X})}$ has already been proved to be an equivalence in the neutral case, that is $\mathcal{X} = \mathrm{B}_R G$, where G is a flat and affine group scheme over R (see [Bro13, Theorem 1.2], where R is a Dedekind domain, and [Sch13, Theorem 1.3.2] for general rings R), for particular quotient stacks over a field (see [Sav06] and 2.10) and for quasi-compact and quasi-separated schemes (see [BC14, Proposition 1.8]). More generally, although not explicitly stated elsewhere, for stacks with the resolution property and with affine diagonal (which is automatic in the algebraic case, see [Tot02]) the fact that $\mathcal{P}_{\mathrm{Vect}(\mathcal{X})}$ is an equivalence is equivalent to the known analogous results where Vect is replaced by $\mathrm{QCoh}(-)$ (for those results and other variants with $\mathrm{Coh}(-)$ or the derived category $D(-)$ see [Lur04, Sch12, BC14, Bra14, Bha16, HR14]): one can pass from quasi-coherent sheaves to locally free sheaves via dualizable objects and, for the converse, extend functors from $\mathrm{Vect}(-)$ to $\mathrm{QCoh}(-)$ following the proof of [Bha16, Corollary 3.2]. We complete this picture by showing that in general the resolution property implies the affineness of the diagonal (see 2.9).

The proof we present here does not follow this strategy and one of the main ingredients is the classification of quasi-compact stacks whose quasi-coherent sheaves are generated by global sections, called pseudo-affine. We show that a quasi-compact stack \mathcal{X} is pseudo-affine if and only if it is a sheaf with a flat monomorphism $\mathcal{X} \rightarrow \mathrm{Spec} B$ for some B (one can take $B = \mathrm{H}^0(\mathcal{O}_{\mathcal{X}})$, see 2.2). Moreover we show that a pseudo-affine algebraic stack is quasi-affine (see 2.8), which has already been observed in [Gro13, Proposition 3.1], and that a quasi-compact flat monomorphism of algebraic stacks is quasi-affine (see 2.7), which was proved in [Ray67].

The characterization of pseudo-affine sheaves and Theorem A are a consequence of the theory developed in [Ton20], where a correspondence between linear functors $\mathcal{C} \rightarrow \mathrm{Mod}(A)$ and quasi-coherent sheaves on $\mathcal{X} \times A$ is discussed. We summarize in Section 1 the results used.

As explained above pseudo-affine sheaves are used in the proof of Theorem A. On the other hand, another consequence of Theorem A is a different characterization of pseudo-affine sheaves: we show that a sheaf U is pseudo-affine if and only if it is the (sheaf) intersection of quasi-compact open subsets of an affine scheme $\text{Spec } B$ (one can take $B = H^0(\mathcal{O}_U)$, see 4.2).

A consequence of this new characterization is a partial answer to our second initial question, the nature of $\text{Fib}_{\mathcal{X},\mathcal{C}}$, and a partial converse to Theorem A.

Theorem B (5.1). *Let \mathcal{X} be a quasi-compact fibered category over R and $\mathcal{C} \subseteq \text{Vect } \mathcal{X}$ be a full monoidal subcategory with duals. If R is not a \mathbb{Q} -algebra assume moreover that $\text{Sym}^n \mathcal{E} \in \mathcal{C}$ for all $\mathcal{E} \in \mathcal{C}$ and $n \in \mathbb{N}$. Then $\text{Fib}_{\mathcal{X},\mathcal{C}}$ is a quasi-compact fpqc stack with affine diagonal and the subcategory $\{\mathcal{G}_{\mathcal{E}}\}_{\mathcal{E} \in \mathcal{C}} \subseteq \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}})$ generates $\text{QCoh}(\text{Fib}_{\mathcal{X},\mathcal{C}})$. In particular $\text{Fib}_{\mathcal{X},\mathcal{C}}$ has the resolution property.*

We also consider the substack $\text{Fib}_{\mathcal{X},\mathcal{C}}^f$ of $\text{Fib}_{\mathcal{X},\mathcal{C}}$ of functors $\Gamma: \mathcal{C} \rightarrow \text{Vect}(A)$ such that $f(\mathcal{E}) = \text{rk } \Gamma_{\mathcal{E}}$ for all $\mathcal{E} \in \mathcal{C}$, where f is a rank function $f: \mathcal{C} \rightarrow \mathbb{N}$. In Theorem 5.1 we prove that $\text{Fib}_{\mathcal{X},\mathcal{C}}^f$ is non empty if and only if there exists a geometric point $\xi: \text{Spec } \Omega \rightarrow \mathcal{X}$ such that $f(\mathcal{E}) = \text{rk } \xi^* \mathcal{E}$ for all $\mathcal{E} \in \mathcal{C}$ and that, in this case, $\text{Fib}_{\mathcal{X},\mathcal{C}}^f$ is also a quasi-compact stack with affine diagonal and the resolution property.

In Theorem 5.1 is also discussed a criterion to deduce that $\text{Fib}_{\mathcal{X},\mathcal{C}}$ is an algebraic stack, but we still think this is very unsatisfactory (see 5.2).

We expect that $\mathcal{G}: \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}})$ is fully faithful in general, which would imply that $\mathcal{X} \rightarrow \text{Fib}_{\mathcal{X},\text{Vect}(\mathcal{X})}$ is universal among maps from \mathcal{X} to quasi-compact fpqc stacks with quasi-affine diagonal and the resolution property, but we are unable to prove it (see 5.19).

We conclude the paper by looking at the “baby case” $\mathcal{C} = \{\mathcal{O}_{\mathcal{X}}\}$. In this case $\text{Fib}_{\mathcal{X},\{\mathcal{O}_{\mathcal{X}}\}}$ is the pseudo-affine sheaf intersection of all quasi-compact open subsets of $\text{Spec } H^0(\mathcal{O}_{\mathcal{X}})$ containing the image of $\mathcal{X} \rightarrow \text{Spec } H^0(\mathcal{O}_{\mathcal{X}})$ (see 5.20). In 5.22 we show that $\text{Fib}_{\mathcal{X},\{\mathcal{O}_{\mathcal{X}}\}}$ is not algebraic in general, even if \mathcal{X} is an integral scheme of finite type over a field.

The outline of the paper is the following. In the first section we describe the theory of sheafification functors from [Ton20] and some results about frame bundles. In the second section we give a first description of pseudo-affine stacks and, in the third section, prove Theorem A. In the fourth section we give a different description of pseudo-affine stacks, which is then used in the last section to prove Theorem B.

NOTATION

In this paper we work over a base commutative, associative ring R with unity. If not stated otherwise a fiber category will be a category fibered in groupoids over Aff/R , the category of affine schemes over $\text{Spec } R$, or, equivalently, the opposite of the category of R -algebras. An fpqc stack will be a stack for the fpqc topology.

A map $f: \mathcal{X}' \rightarrow \mathcal{X}$ of fibered categories is called representable if for all maps $T \rightarrow \mathcal{X}$ from an affine scheme (or an algebraic space) the fiber product $T \times_{\mathcal{X}} \mathcal{X}'$ is (equivalent to) an algebraic space.

Given a flat and affine group scheme G over R we denote by $B_R G$ (or simply $B G$ when the base ring is clear) the stack of G -torsors for the fpqc topology, which is an fpqc stack with affine diagonal. When $G \rightarrow \text{Spec } R$ is finitely presented (resp. smooth) then $B_R G$ coincides with the stack of G -torsors for the fppf (resp. étale) topology.

By a “subcategory” of a given category we mean a “full subcategory” if not stated otherwise.

We assume the notations, definition and results of [Ton20, Section 1]. In particular: the notion of pseudo-algebraic and quasi-compact fibered categories or maps between them; flat maps of fibered categories; quasi-coherent sheaves and their functoriality; quasi-coherent sheaves on a

fibered category. In particular $\mathrm{QCoh}(\mathcal{X})$ and $\mathrm{QAlg}(\mathcal{X})$ denotes the category of quasi-coherent sheaves and quasi-coherent algebras on \mathcal{X} respectively.

If \mathcal{Y}_i for $i \in I$ is a set of fibered categories over R then $\sqcup_i \mathcal{Y}_i$ is defined as the fibered category over R whose objects over an R -algebra A are tuples consisting of a decomposition $\mathrm{Spec} A = \sqcup_i U_i$ into open and closed subsets and $y_i \in \mathcal{Y}_i(U_i)$. In particular if \mathcal{X} is a fibered category with maps $\mathcal{Y}_i \rightarrow \mathcal{X}$ then a map

$$\bigsqcup_{i \in I} \mathcal{Y}_i \rightarrow \mathcal{X}$$

is well defined provided that \mathcal{X} is a Zariski stack.

If \mathcal{E} is a vector bundle over a category fibered in groupoids \mathcal{X} then the locus $\mathcal{X}_n \rightarrow \mathcal{X}$ where \mathcal{E} has rank n is an open and closed immersion. Moreover there is a map $\mathcal{X} \rightarrow \sqcup_n \mathcal{X}_n$ (which is an equivalence if \mathcal{X} is a Zariski stack) and if \mathcal{X} is quasi-compact, then all \mathcal{X}_n are empty but finitely many.

If \mathcal{E} is a vector bundle on a fibered category \mathcal{X} we can define $\det \mathcal{E}$ as

$$\det \mathcal{E} = \left(\bigoplus_{n \in \mathbb{N}} \Lambda^n(\mathcal{E}|_{\mathcal{X}_n}) \right) |_{\mathcal{X}}$$

which is compatible with the usual notion of determinant for vector bundles of fixed rank.

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1. PRELIMINARIES

We recall some basic definitions and set up some notations. We fix a base ring R .

1.1. Monoidal functors and their sheafification.

Definition 1.1. Let \mathcal{C} and \mathcal{D} be R -linear symmetric monoidal categories. A (contravariant) *pseudo-monoidal* functor $\Omega: \mathcal{C} \rightarrow \mathcal{D}$ is an R -linear (and contravariant) functor together with a natural transformation

$$\iota_{V,W}^\Omega: \Omega_V \otimes \Omega_W \rightarrow \Omega_{V \otimes W} \text{ for } V, W \in \mathcal{C}$$

A (contravariant) pseudo-monoidal functor $\Omega: \mathcal{C} \rightarrow \mathcal{D}$ is

- 1) *symmetric* or commutative if for all $V, W \in \mathcal{C}$ the following diagram is commutative

$$\begin{array}{ccc} \Omega_V \otimes \Omega_W & \xrightarrow{\iota_{V,W}^\Omega} & \Omega_{V \otimes W} \\ \downarrow & & \downarrow \\ \Omega_W \otimes \Omega_V & \xrightarrow{\iota_{W,V}^\Omega} & \Omega_{W \otimes V} \end{array}$$

where the vertical arrows are the obvious isomorphisms;

2) *associative* if for all $V, W, Z \in \mathcal{C}$ the following diagram is commutative

$$\begin{array}{ccc} \Omega_V \otimes \Omega_W \otimes \Omega_Z & \xrightarrow{\iota_{V,W}^\Omega \otimes \text{id}} & \Omega_{V \otimes W} \otimes \Omega_Z \\ \downarrow \text{id} \otimes \iota_{W,Z}^\Omega & & \downarrow \iota_{V \otimes W, Z}^\Omega \\ \Omega_V \otimes \Omega_{W \otimes Z} & \xrightarrow{\iota_{V,W \otimes Z}^\Omega} & \Omega_{V \otimes W \otimes Z} \end{array}$$

If I and J are the unit objects of \mathcal{C} and \mathcal{D} respectively, a unity for Ω is a morphism $1: J \rightarrow \Omega_I$ such that, for all $V \in \mathcal{C}$, the compositions

$$\Omega_V \otimes J \rightarrow \Omega_V \otimes \Omega_I \rightarrow \Omega_{V \otimes I} \rightarrow \Omega_V \quad \text{and} \quad J \otimes \Omega_V \rightarrow \Omega_I \otimes \Omega_V \rightarrow \Omega_{I \otimes V} \rightarrow \Omega_V$$

coincide with the natural isomorphisms $\Omega_V \otimes J \rightarrow \Omega_V$ and $J \otimes \Omega_V \rightarrow \Omega_V$ respectively.

A (contravariant) *monoidal* functor $\Omega: \mathcal{C} \rightarrow \mathcal{D}$ is a symmetric and associative pseudo-monoidal (contravariant) functor with a unity 1. A (contravariant) *strong* monoidal functor $\Omega: \mathcal{C} \rightarrow \mathcal{D}$ is a (contravariant) monoidal functors such that all the maps $\iota_{V,W}^\Omega$ and $1: J \rightarrow \Omega_I$ are isomorphisms.

A morphism of pseudo-monoidal functors $(\Omega, \iota^\Omega) \rightarrow (\Gamma, \iota^\Gamma)$, called a monoidal morphism or transformation, is a natural transformation $\Omega \rightarrow \Gamma$ which commutes with the monoidal structures ι^* . A morphism of monoidal functors is a monoidal transformation preserving the unities.

Definition 1.2. Let \mathcal{X} be a fibered category over R and \mathcal{C} a full subcategory of $\text{QCoh}(\mathcal{X})$. We say that \mathcal{C} generates $\text{QCoh}(\mathcal{X})$ if all quasi-coherent sheaves on \mathcal{X} are a quotient of a direct sum of sheaves in \mathcal{C} . We say that \mathcal{X} has the *resolution property* if $\text{Vect}(\mathcal{X})$ generates $\text{QCoh}(\mathcal{X})$.

We will consider only fpqc stacks with quasi-affine diagonal, for instance algebraic stacks with quasi-affine diagonal (see [LMB05, Corollary 10.7]) and quasi-separated schemes. This is because resolution property is somehow meaningless for other stacks, see for instance Remark (1) in the introduction of [Tot02].

Definition 1.3. [Ton20, Def 2.1] Let \mathcal{X} be a fibered category over R , A an R -algebra and \mathcal{D} a subcategory $\mathcal{D} \subseteq \text{QCoh}(\mathcal{X})$.

We define $\text{L}_R(\mathcal{D}, A)$ as the category of contravariant R -linear functors $\Gamma: \mathcal{D} \rightarrow \text{Mod } A$ and natural transformations as arrows. If \mathcal{D} is a monoidal subcategory of $\text{QCoh} \mathcal{X}$, that is a subcategory such that $\mathcal{O}_{\mathcal{X}} \in \mathcal{D}$ and for all $\mathcal{E}, \mathcal{E}' \in \mathcal{D}$ we have $\mathcal{E} \otimes \mathcal{E}' \in \mathcal{D}$, we also define the category $\text{ML}_R(\mathcal{D}, A)$ whose objects are $\Gamma \in \text{L}_R(\mathcal{D}, A)$ with a monoidal structure.

We denote by \mathcal{D}^\oplus the full subcategory of $\text{QCoh}(\mathcal{X})$ containing all the finite direct sums of elements in \mathcal{D} . Notice that a (contravariant) R -linear functor from \mathcal{D} to an R -linear and additive category extends uniquely to \mathcal{D}^\oplus (see [Ton20, Prop 2.16]). We will denote this extension by the same symbol. In other words we will evaluate a linear functor with source \mathcal{D} also on objects and maps of \mathcal{D}^\oplus .

Definition 1.4. A *finite test sequence* for \mathcal{D} is an exact sequence in $\text{QCoh}(\mathcal{X})$ of the form

$$\mathcal{E}' \rightarrow \mathcal{E} \rightarrow 0 \quad \text{or} \quad \mathcal{E}'' \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow 0 \quad \text{with} \quad \mathcal{E}'', \mathcal{E}', \mathcal{E} \in \mathcal{D}^\oplus$$

If $\mathcal{D} \subseteq \text{Vect}(\mathcal{X})$, \mathcal{X} is quasi-compact and \mathcal{Y} is another fiber category over R we say that a (contravariant) R -linear functor $\Gamma: \mathcal{D} \rightarrow \text{QCoh}(\mathcal{Y})$ is right (resp. left) exact if it is exact on all test sequences in \mathcal{D} . We define $\text{Lex}_R(\mathcal{D}, A)$ (resp. $\text{MLex}_R(\mathcal{D}, A)$) as the subcategory of $\text{L}_R(\mathcal{D}, A)$ (resp. $\text{ML}_R(\mathcal{D}, A)$) of left exact functors.

We define $\mathcal{X}_A = \mathcal{X} \times_R A$ and

$$\Omega^*: \text{QCoh} \mathcal{X}_A \rightarrow \text{L}_R(\mathcal{D}, A), \quad \Omega_{\mathcal{H}}^{\mathcal{G}} = \text{Hom}_{\mathcal{X}}(\pi^* \mathcal{H}, \mathcal{G})$$

where $\pi: \mathcal{X}_A \rightarrow \mathcal{X}$ is the projection.

Remark 1.5. The definition of test sequence and therefore of the categories $\text{Lex}_R(\mathcal{D}, A)$ and $\text{MLex}_R(\mathcal{D}, A)$ it is not completely equivalent to the one in [Ton20, Def 4.3]. As discussed in the proof of the next Theorem the two notions agree when \mathcal{D} generates $\text{QCoh}(\mathcal{X})$: functors left exact in the sense of [Ton20] are automatically left exact in our sense. In [Ton20] we were looking for a minimal collection of test sequences which make results like the one below true under the assumption that \mathcal{D} generates $\text{QCoh}(\mathcal{X})$. In this paper instead we will deal with more general subcategories \mathcal{D} and therefore this notion of test sequence seems more precise.

Theorem 1.6. *Let \mathcal{X} be a quasi-compact fibered category over R and $\mathcal{D} \subseteq \text{Vect}(\mathcal{X})$ be a full subcategory that generates $\text{QCoh}(\mathcal{X})$. Then $\Omega^*: \text{QCoh} \mathcal{X}_A \rightarrow \text{L}_R(\mathcal{D}, A)$ is fully faithful with essential image $\text{Lex}_R(\mathcal{D}, A)$ and it has an exact left adjoint $\mathcal{F}_{*, \mathcal{D}}: \text{L}_R(\mathcal{D}, A) \rightarrow \text{QCoh} \mathcal{X}_A$. In particular Ω_* and $(\mathcal{F}_{*, \mathcal{D}})|_{\text{Lex}_R(\mathcal{D}, A)}$ are quasi-inverses of each other.*

Assume that \mathcal{D} is a monoidal subcategory of $\text{Vect}(\mathcal{X})$. Then the functor Ω^ and $\mathcal{F}_{*, \mathcal{D}}$ extends to adjoint functors $\Omega^*: \text{QAlg} \mathcal{X}_A \rightarrow \text{ML}_R(\mathcal{D}, A)$ and $\mathcal{A}_{*, \mathcal{C}}: \text{ML}_R(\mathcal{D}, A) \rightarrow \text{QAlg} \mathcal{X}_A$. Moreover $\Omega^*: \text{QAlg}(\mathcal{X}_A) \rightarrow \text{MLex}_R(\mathcal{D}, A)$ is an equivalence and $(\mathcal{A}_{*, \mathcal{C}})|_{\text{MLex}_R(\mathcal{D}, A)}$ is a quasi-inverse.*

Proof. By [Ton20, Theorem A], [Ton20, Theorem B] and [Ton20, Theorem 6.8] the same result works for the notion of test sequence given in [Ton20, Def 2.1]. On the other hand by [Ton20, Prop 4.5] functors in the image of $\Omega^*: \text{QCoh} \mathcal{X}_A \rightarrow \text{L}_R(\mathcal{D}, A)$ are automatically left exact in our stronger sense. \square

Remark 1.7. If \mathcal{X} is a quasi-compact fibered category over R and $\mathcal{C} \subseteq \text{Vect}(\mathcal{X})$ is a full (monoidal) subcategory there always exists a full (monoidal) subcategory $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ such that $\tilde{\mathcal{C}}$ is small, that is the class $\text{ob}(\tilde{\mathcal{C}})$ is a set, and $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ is an equivalence. In particular in this case the restriction along $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ induces an equivalence for all the categories of linear (monoidal) functors we have considered. Thus, when \mathcal{X} is a quasi-compact fibered category over R , we will tacitly assume that the category \mathcal{C} is small.

To see how to find $\tilde{\mathcal{C}}$ we use the following argument. The category \mathcal{C} is essentially small because \mathcal{X} is quasi-compact (see [Ton20, Prop 1.8]), so we can start taking $\mathcal{R}_0 \subseteq \text{ob}(\mathcal{C})$ so that any sheaf in \mathcal{C} is isomorphic to a sheaf in \mathcal{R}_0 . If we don't need duals or monoidal structure we can stop here and consider as $\tilde{\mathcal{C}}$ the full subcategory with objects in \mathcal{R}_0 .

If \mathcal{C} is a submonoidal category, this choice is not enough, since we don't want to change the \otimes . Assume also $\mathcal{O}_{\mathcal{X}} \in \mathcal{R}_0$. At this point we denote by \mathcal{R} the set of objects of \mathcal{C} that can be written using finitely many times \otimes and $-^\vee$, starting from objects in \mathcal{R}_0 . Setting as $\tilde{\mathcal{C}}$ the full subcategory of \mathcal{C} with objects in \mathcal{R} makes the trick.

1.2. Generalized frame bundles. In this section we study frame bundles of general vector bundles. We fix a category \mathcal{X} fibered in groupoids over R .

Definition 1.8. Given a subset $K \subseteq \mathbb{N}$ we set

$$D(K) = \bigsqcup_{n \in K} \text{Spec } R$$

and denote by $Q(K)$ the vector bundle on $D(K)$ which is free of rank n over the copy of $\text{Spec } R$ corresponding to $n \in K$. More generally given a collection $K_*: I \rightarrow \mathcal{P}(\mathbb{N})$, where $\mathcal{P}(\mathbb{N})$ is the set of subsets of \mathbb{N} , we set $D(K_*) = \prod_i D(K_i)$ as a sheaf over R .

Let $\mathcal{E} \in \text{Vect}(\mathcal{X})$ be a locally free sheaf on \mathcal{X} . We set

$$\text{ranks}(\mathcal{E}) = \{n \in \mathbb{N} \mid \exists s: \text{Spec } L \rightarrow \mathcal{X} \text{ with } \text{rk } s^* \mathcal{E} = n\}, \quad D(\mathcal{E}) = D(\text{ranks}(\mathcal{E})), \quad Q(\mathcal{E}) = Q(\text{ranks}(\mathcal{E}))$$

There is a map $r_{\mathcal{E}}: \mathcal{X} \rightarrow D(\mathcal{E})$, the rank function, and we set

$$\text{Fr}(\mathcal{E}) = \underline{\text{Iso}}_{\mathcal{X}}(\mathcal{E}, r_{\mathcal{E}}^* Q(\mathcal{E})) \rightarrow \mathcal{X}$$

If I is a set and $\mathcal{E}_*: I \rightarrow \text{Vect}(\mathcal{X})$ is a collection of vector bundles we set $D(\mathcal{E}_*) = D(\text{ranks}(\mathcal{E}_*)) = \prod_i D(\mathcal{E}_i)$ and $\text{Fr}(\mathcal{E}_*) = \prod_i \text{Fr}(\mathcal{E}_i)$, where the products are taken over R and \mathcal{X} respectively.

Remark 1.9. The scheme $D(K)$, thought of as a sheaf, is the constant sheaf associated with K . Instead $D(K_*)$ as sheaf $\text{Aff}/R \rightarrow (\text{Sets})$ is

$$D(K_*)(U) = \{s: U \rightarrow \prod_{i \in I} K_i \text{ with locally constant factors } U \rightarrow K_i\}$$

The map $\mathcal{X} \rightarrow D(\mathcal{E}_*)$ maps $\xi: U \rightarrow \mathcal{X}$ to the map $\prod_i \text{rk}(\xi^* \mathcal{E}_i): U \rightarrow \prod_{i \in I} \text{ranks}(\mathcal{E}_i)$.

Remark 1.10. The scheme $D(K)$ is affine if and only if K is finite. If $K_*: I \rightarrow \mathcal{P}(\mathbb{N})$ is a collection such that all the K_i are finite, then $D(K_*) \rightarrow \text{Spec } R$ is also affine and faithfully flat. This is the case, for instance, if \mathcal{X} is quasi-compact and $K_* = \text{ranks}(\mathcal{E}_*)$. Otherwise $D(K_*)$ is just thought of as a sheaf over R .

Lemma 1.11. *Let $K_*: I \rightarrow \mathcal{P}(\mathbb{N})$ be a map and $n_i \in K_i$ for all i . Then the map $\text{Spec } R \rightarrow D(K_*)$ given by the constant function $(n_i)_i: \text{Spec } R \rightarrow \prod_{i \in I} K_i$ is a closed immersion and representable by localizations of affine schemes. In particular if $\mathcal{E}_*: I \rightarrow \text{Vect}(\mathcal{X})$ is a collection of vector bundles then the locus $\mathcal{X}' \rightarrow \mathcal{X}$ where the sheaves \mathcal{E}_i have rank n_i enjoys the same property.*

Proof. Let $\text{Spec } B \rightarrow D(K_*)$ be any map, given by locally constant functions $h_i: \text{Spec } B \rightarrow K_i$. Set $Z = \text{Spec } B \times_{D(K_*)} \text{Spec } R$. We have to show that $Z \rightarrow \text{Spec } B$ is a closed immersion and a localization. By definition of fiber product we see that Z is the locus in $\text{Spec } B$ where the h_i are constant and equal to n_i . In other words if $U_i = h_i^{-1}(n_i)$ then $Z = \cap_i U_i$. Notice that any intersections of the U_i is an open and closed subscheme of $\text{Spec } B$ and, in particular, a localization. Since Z can be written as a filtered intersection of open and closed subsets of $\text{Spec } B$, it is elementary to check that $Z \rightarrow \text{Spec } B$ is both a closed immersion and a localization. \square

Proposition 1.12. *Let $K \subseteq \mathbb{N}$. The stack $\text{B}_{D(K)} \text{GL}(Q(K)) \rightarrow D(K)$ is equivalent to the stack over $D(K)$ whose fiber over a $\text{Spec } B \rightarrow D(K)$, given by $h: \text{Spec } B \rightarrow K$, is the groupoid of locally free sheaves on $\text{Spec } B$ that have rank n on $h^{-1}(n)$ for all $n \in K$.*

Let $K_*: I \rightarrow \mathcal{P}(\mathbb{N})$ and

$$G = \prod_i \text{GL}(Q(K_i)) = \prod_i (\text{GL}(Q(K_i)) \times_{D(K_i)} D(K_*)) \rightarrow D(K_*)$$

where the products are taken over R and $D(K_*)$ respectively. Then $G \rightarrow D(K_*)$ is an affine and faithfully flat relative group scheme and the stack $\text{B}_{D(K_*)} G \rightarrow D(K_*)$ is equivalent to the stack over $D(K_*)$ whose fiber over a $\text{Spec } B \rightarrow D(K_*)$, given by $h_i: \text{Spec } B \rightarrow K_i$, is the groupoid of $(\mathcal{G}_i)_i$ where \mathcal{G}_i is a locally free sheaf on $\text{Spec } B$ that have rank n on $h^{-1}(n)$ for all $n \in K_i$.

Proof. The first statement reduced easily to the case of GL_n . For the second part set $G_i = \text{GL}(Q(K_i)) \times_{D(K_i)} D(K_*)$. Everything follows because $G = \prod_i G_i$ and therefore

$$\text{B}_{D(K_*)} G = \prod_i \text{B}_{D(K_*)} G_i$$

where the products are taken over $D(K_*)$. \square

Lemma 1.13. *Let $\mathcal{E} \in \text{Vect}(\mathcal{X})$. Then there is a Cartesian diagram*

$$\begin{array}{ccc} \text{Fr}(\mathcal{E}) & \longrightarrow & D(\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \text{B}_{D(\mathcal{E})}(\text{GL}(Q(\mathcal{E}))) \end{array}$$

where $\mathcal{X} \rightarrow \mathrm{B}_{D(\mathcal{E})} \mathrm{GL}(Q(\mathcal{E}))$ is induced by \mathcal{E} as in 1.12. In particular $\mathrm{Fr}(\mathcal{E}) \rightarrow \mathcal{X}$ is a torsor under the affine group scheme $\mathrm{GL}(Q(\mathcal{E})) \rightarrow D(\mathcal{E})$ and it is affine, smooth and surjective.

Let $\mathcal{E}_*: I \rightarrow \mathrm{Vect}(\mathcal{X})$ be a collection of sheaves and consider $G = \prod_i \mathrm{GL}(Q(\mathcal{E}_i)) \rightarrow \prod_i D(Q(\mathcal{E}_i)) = D(\mathcal{E}_*)$. Then there is a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Fr}(\mathcal{E}_*) & \longrightarrow & D(\mathcal{E}_*) \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathrm{B}_{D(\mathcal{E}_*)}(G) \end{array}$$

where $\mathcal{X} \rightarrow \mathrm{B}_{D(\mathcal{E}_*)} G$ is induced by the $(\mathcal{E}_i)_i$ as in 1.12. In particular $\mathrm{Fr}(\mathcal{E}_*) \rightarrow \mathcal{X}$ is affine and faithfully flat as well.

Proof. It follows from the definition of Fr and the description in 1.12. \square

Lemma 1.14. *Let \mathcal{E} be a locally free sheaf on \mathcal{X} of rank n and set V for the free R -module of rank n with basis v_1, \dots, v_n . Then the map*

$$\gamma: \det \mathcal{E} \rightarrow \mathrm{Sym}^n(V \otimes \mathcal{E}), \quad \gamma(x_1 \wedge \cdots \wedge x_n) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_k (v_{\sigma(k)} \otimes x_k)$$

is well defined and has locally free cokernel. In particular its dual $\mathrm{Sym}^n(V \otimes \mathcal{E})^\vee \rightarrow \det \mathcal{E}^\vee$ is surjective. Moreover, if \mathcal{E} is free with basis e_1, \dots, e_n then $e_1 \wedge \cdots \wedge e_n \in \det \mathcal{E}$ is sent to the determinant of the matrix $(v_i \otimes e_j)_{ij}$, which lies in $\mathrm{Sym}^n(V \otimes \mathcal{E})$.

Proof. Consider the lift $\tilde{\gamma}: \mathcal{E}^{\otimes n} \rightarrow (V \otimes \mathcal{E})^{\otimes n} \rightarrow \mathrm{Sym}^n(V \otimes \mathcal{E})$ of γ . This is certainly well defined. In order to show that it factors through the quotient $\det \mathcal{E}$, so that γ would be well defined, we can assume that \mathcal{E} is free with basis e_1, \dots, e_n . Set $y_{ij} = v_i \otimes e_j \in V \otimes \mathcal{E}$ and $e_\omega = e_{\omega(1)} \otimes \cdots \otimes e_{\omega(n)} \in \mathcal{E}^{\otimes n}$ for any map $\omega: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$. We have

$$\tilde{\gamma}(e_\omega) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_k y_{\sigma(k)\omega(k)} = \det(y_{i,\omega(j)})$$

It follows that $\tilde{\gamma}(e_\omega) = 0$ if ω is not a permutation, that $\tilde{\gamma}(e_{\tau\omega}) = -\tilde{\gamma}(e_\omega)$ if τ is a transposition and that $\tilde{\gamma}(e_{\mathrm{id}}) = \det(v_i \otimes e_j)$, as required.

We now show that the cokernel of γ is locally free. This is equivalent to the surjectivity of $\mathrm{Sym}^n(V \otimes \mathcal{E})^\vee \rightarrow \det \mathcal{E}^\vee$ and it can be checked locally. Consider $e: \mathrm{Sym}^n(V \otimes \mathcal{E}) \rightarrow R$ given by $e(y_{ij}) = \delta_{ij}$, the Kronecker symbol. We see that the composition $\det \mathcal{E} \rightarrow \mathrm{Sym}^n(V \otimes \mathcal{E}) \xrightarrow{e} R$ maps $e_1 \wedge \cdots \wedge e_n \rightarrow \det(I) = 1$, where $I = (\delta_{ij})$ is the identity matrix. \square

Lemma 1.15. *Let \mathcal{E} be a vector bundle on a fibered category \mathcal{X} over R then:*

- 1) *the sheaf $\det \mathcal{E}$ is a quotient of a direct sum of the $\mathcal{E}^{\otimes m}$ for $m \in \mathbb{N}$;*
- 2) *if \mathcal{X} is defined over \mathbb{Q} then $(\mathrm{Sym}^q \mathcal{E})^\vee$ is a quotient of $(\mathcal{E}^\vee)^{\otimes q}$.*
- 3) *the sheaf $\mathrm{Sym}^q(\mathcal{E} \otimes M)^\vee$, for a locally free R -module M , is a quotient of a direct sum of finite tensor products of the $\mathrm{Sym}^a(\mathcal{E})^\vee$ for $a \in \mathbb{N}$.*

Proof. We can assume that \mathcal{X} is a Zariski stack. Set $I = \mathrm{ranks}(\mathcal{E})$ and denote by \mathcal{X}_n for $n \in I$ the locus in \mathcal{X} where \mathcal{E} has rank n , so that $\sqcup_n \mathcal{X}_n = \mathcal{X}$.

- 1). There are surjective maps

$$\bigoplus_{n \in I} \mathcal{E}^{\otimes n} \rightarrow \bigoplus_{n \in I} (\mathcal{E}^{\otimes n})|_{\mathcal{X}_n} \rightarrow \bigoplus_{n \in I} \det(\mathcal{E}|_{\mathcal{X}_n}) \simeq \det \mathcal{E}$$

- 2) There is a surjective map

$$(\mathcal{E}^\vee)^{\otimes q} \rightarrow \mathrm{Sym}^q(\mathcal{E}^\vee) \simeq (\mathrm{Sym}^q \mathcal{E})^\vee$$

3) Write $M \oplus N = R^l$. We use the formula

$$\mathrm{Sym}^q(A \oplus B) = \bigoplus_{u+v=q} \mathrm{Sym}^u(A) \otimes \mathrm{Sym}^v(B)$$

We have

$$\mathrm{Sym}^q(\mathcal{E} \otimes R^l) = \mathrm{Sym}^q(\mathcal{E} \otimes M) \oplus P \implies \mathrm{Sym}^q(\mathcal{E} \otimes R^l)^\vee \twoheadrightarrow \mathrm{Sym}^q(\mathcal{E} \otimes M)^\vee$$

Moreover $\mathrm{Sym}^q(\mathcal{E} \otimes R^l)^\vee$ can be written as a quotient of a direct sum of finite tensor products of the $\mathrm{Sym}^a(\mathcal{E})^\vee$ for $a \in \mathbb{N}$. \square

Lemma 1.16. *Let I be a set and $\mathcal{E}_*: I \rightarrow \mathrm{Vect}(R)$ be any map. Set*

$$\mathrm{GL}(\mathcal{E}_*) = \prod_{i \in I} \mathrm{GL}(\mathcal{E}_i) \rightarrow \mathrm{Spec} R$$

and \mathcal{F}_i for the locally free sheaf on $\mathrm{BGL}(\mathcal{E}_*)$ coming from the universal one on $\mathrm{BGL}(\mathcal{E}_i)$, that is the R -module \mathcal{E}_i with the natural action of $\mathrm{GL}(\mathcal{E}_i)$. Consider the following sets:

$$\mathcal{R}_1 = \{\det \mathcal{F}_i, (\mathrm{Sym}^m \mathcal{F}_i)^\vee\}_{i \in I, m \in \mathbb{N}}, \quad \mathcal{R}_2 = \{\mathcal{F}_i, (\mathrm{Sym}^m \mathcal{F}_i)^\vee\}_{i \in I, m \in \mathbb{N}}, \quad \mathcal{R}_3 = \{\mathcal{F}_i, \mathcal{F}_i^\vee\}_{i \in I}$$

Then the subcategory of $\mathrm{Vect}(\mathrm{BGL}(\mathcal{E}_*))$ consisting of all tensor products of sheaves in \mathcal{R}_1 (resp. \mathcal{R}_2) generates $\mathrm{QCoh}(\mathrm{BGL}(\mathcal{E}_*))$.

If R is a \mathbb{Q} -algebra then we can replace $(\mathrm{Sym}^m \mathcal{F}_i)^\vee$ by \mathcal{F}_i^\vee in the statement above.

Proof. Denote by $\mathcal{C} \subseteq \mathrm{QCoh}(\mathrm{BGL}(\mathcal{E}_*))$ the full subcategory generated by the sheaves \mathcal{R}_i in the statement (here i can be 1, 2, 3 depending on the hypothesis). We have to show that \mathcal{C} generates $\mathrm{QCoh}(\mathrm{BGL}(\mathcal{E}_*))$, so that, a posteriori, $\mathcal{C} = \mathrm{QCoh}(\mathrm{BGL}(\mathcal{E}_*))$. We claim that $R[\mathrm{GL}(\mathcal{E}_*)]$ is a direct limit of representations whose dual belongs to \mathcal{C} . By [Ton20, Prop 8.2] this will end the proof.

The representation $R[\mathrm{GL}(\mathcal{E}_*)]$ is a direct limit of tensor products of the regular representations $R[\mathrm{GL}(\mathcal{E}_i)]$ for $i \in I$. This allows us to reduce to the case of $\mathrm{GL}(\mathcal{E})$ when \mathcal{E} is a vector bundle on \mathcal{X} (i.e. I has one element). Call \mathcal{F} the universal locally free sheaf on $\mathrm{BGL}(\mathcal{E})$, that is $\mathcal{F} = \mathcal{E}$ with the action of $\mathrm{GL}(\mathcal{E})$. Recall that for any locally free representation \mathcal{H} of $R[\mathrm{GL}(\mathcal{E})]$ there is a canonical isomorphism of R -modules $\mathrm{Hom}^{\mathrm{GL}(\mathcal{E})}(\mathcal{H}, R[\mathrm{GL}(\mathcal{E})]) \simeq \mathcal{H}^\vee$ (see [Ton20, Rem 8.1]) So there are equivariant maps

$$\mathcal{F} \otimes \mathrm{Hom}^{\mathrm{GL}(\mathcal{E})}(\mathcal{F}, R[\mathrm{GL}(\mathcal{E})]) \simeq \mathcal{F} \otimes \mathcal{E}^\vee \rightarrow R[\mathrm{GL}(\mathcal{E})]$$

and, for $m \in \mathbb{N}$,

$$(\det \mathcal{F})^{-\otimes m} \otimes \mathrm{Hom}^{\mathrm{GL}(\mathcal{E})}((\det \mathcal{F})^{-\otimes m}, R[\mathrm{GL}(\mathcal{E})]) \simeq (\det \mathcal{F})^{-\otimes m} \otimes (\det \mathcal{E})^{\otimes m} \rightarrow R[\mathrm{GL}(\mathcal{E})]$$

Consider the equivariant maps

$$A_m = \left[\bigoplus_{q=0}^{2mn} \mathrm{Sym}^q(\mathcal{F} \otimes \mathcal{E}^\vee) \right] \otimes [(\det \mathcal{F})^{-\otimes m} \otimes (\det \mathcal{E})^{\otimes m}] \longrightarrow R[\mathrm{GL}(\mathcal{E})] \otimes R[\mathrm{GL}(\mathcal{E})] \rightarrow R[\mathrm{GL}(\mathcal{E})]$$

We claim that those maps are injective and their images form an increasing sequence of subrepresentations saturating $R[\mathrm{GL}(\mathcal{E})]$. This statement is local, so we can assume $\mathcal{E} = \mathcal{O}_{\mathcal{X}}^n$, so that $\mathrm{GL}(\mathcal{E}) = \mathrm{GL}_n$. As usual we can write $R[\mathrm{GL}_n] = R[X_{uv}]_{\det}$ for $1 \leq u, v \leq n$, where \det is the determinant polynomial. In this case $\mathcal{F} \otimes \mathcal{E}^\vee \rightarrow R[\mathrm{GL}_n]$ is an isomorphism onto the R -submodule generated by all the $X_{u,v}$, while $(\det \mathcal{F})^{-\otimes m} \otimes (\det \mathcal{E})^{\otimes m} \rightarrow R[\mathrm{GL}(\mathcal{E})]$ is an isomorphism onto the R -submodule generated by \det^{-m} . Thus $A_m \rightarrow R[\mathrm{GL}_n]$ is an isomorphism onto the set of fractions f/\det^m where f is a polynomial of degree less or equal to $2nm$. It is now easy to see that $\cup_m A_m = R[\mathrm{GL}_n]$.

We come back to the general setting. We need to show that $A_m^\vee \in \mathcal{C}$ for all m . The last statement follows from the first thanks to 1.15, 2). Since \mathcal{C} is closed under tensor product and direct sum, we have to show that

$$(\mathrm{Sym}^q(\mathcal{F} \otimes \mathcal{E}^\vee))^\vee \text{ for } q > 0, (\det \mathcal{F} \otimes \det \mathcal{E}^{-1}) \in \mathcal{C}$$

For the first sheaf it follows from 1.15, 3). For the second, since $\det \mathcal{E}^{-1}$ is a quotient of a free R -module, it is enough to show that $\det \mathcal{F} \in \mathcal{C}$. For \mathcal{R}_1 is clear. For \mathcal{R}_2 we use 1.15, 1). \square

2. PSEUDO-AFFINE STACKS

One of the key points in the proof of Theorem A is a characterization of the following stacks.

Definition 2.1. A *pseudo-affine* stack is a quasi-compact fpqc stack with quasi-affine diagonal such that all quasi-coherent sheaves on it are generated by global sections. In other words a pseudo-affine stack is a quasi-compact fpqc stack \mathcal{X} such that $\{\mathcal{O}_{\mathcal{X}}\}$ generates $\mathrm{QCoh}(\mathcal{X})$.

A map $f: \mathcal{Y}' \rightarrow \mathcal{Y}$ of fibered categories is called *pseudo-affine* if for all maps $T \rightarrow \mathcal{Y}$ from an affine scheme the fiber product $T \times_{\mathcal{Y}} \mathcal{Y}'$ is pseudo-affine.

A quasi-affine scheme is pseudo-affine. We will prove a pseudo-affine stack which is algebraic is quasi-affine. This result appeared before in [Gro13, Proposition 3.1]. We will show that pseudo-affine stacks are indeed just arbitrary intersection of quasi-compact open subsets of an affine scheme (see 4.2). In general a pseudo-affine sheaf is not quasi-affine. An example is the sheaf intersection of all the complement of closed points in $\mathrm{Spec} k[x, y]$, where k is a field (see 4.3).

We start with a first characterization of pseudo-affine stacks.

Proposition 2.2. *Let $\mathcal{X} \xrightarrow{\pi} \mathrm{Spec} R$ be a quasi-compact fpqc stack with quasi-affine diagonal. Then the following conditions are equivalent:*

- 1) *the stack \mathcal{X} is pseudo-affine;*
- 2) *the map $\pi^* \pi_* \mathcal{F} \rightarrow \mathcal{F}$ is surjective for all $\mathcal{F} \in \mathrm{QCoh} \mathcal{X}$;*
- 3) *the stack \mathcal{X} is equivalent to a sheaf and there exists a flat monomorphism $\mathcal{X} \rightarrow \mathrm{Spec} B$, where B is a ring.*

In this case the map $p: \mathcal{X} \rightarrow \mathrm{Spec} H^0(\mathcal{O}_{\mathcal{X}})$ is a flat monomorphism, $p_: \mathrm{QCoh} \mathcal{X} \rightarrow \mathrm{Mod} H^0(\mathcal{X})$ is fully faithful and $p^* p_* \simeq \mathrm{id}$. Moreover if $\mathcal{X} \times_{H^0(\mathcal{O}_{\mathcal{X}})} k \neq \emptyset$ for all geometric points $\mathrm{Spec} k \rightarrow \mathrm{Spec} H^0(\mathcal{O}_{\mathcal{X}})$ then p is an isomorphism.*

Proof. 2) \implies 1). Given $\mathcal{F} \in \mathrm{QCoh} \mathcal{X}$, take a surjective map $R^{(I)} \rightarrow \pi_* \mathcal{F}$. In this case the composition

$$\mathcal{O}_{\mathcal{X}}^{(I)} \simeq \pi^* R^{(I)} \rightarrow \pi^* \pi_* \mathcal{F} \rightarrow \mathcal{F}$$

is surjective.

1) \implies 2). A sheaf $\mathcal{F} \in \mathrm{QCoh} \mathcal{X}$ is generated by global sections and the image of $\pi^* \pi_* \mathcal{F} \rightarrow \mathcal{F}$ contains all of them.

3) \implies 1). Denote by $p: \mathcal{X} \rightarrow \mathrm{Spec} B$ the flat monomorphism. We are going to show that $\delta_{\mathcal{F}}: p^* p_* \mathcal{F} \rightarrow \mathcal{F}$ is an isomorphism for all $\mathcal{F} \in \mathrm{QCoh} \mathcal{X}$. Arguing as in 2) \implies 1) this will conclude the proof. By hypothesis there exists a representable fpqc covering $h: \mathrm{Spec} C \rightarrow \mathcal{X}$ and we must prove that $h^* \delta_{\mathcal{F}}$ is an isomorphism. Let $f = ph: \mathrm{Spec} C \rightarrow \mathrm{Spec} B$ be the composition and consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec} C & \xrightarrow{s} & \mathcal{X} \times_B C & \xrightarrow{\alpha} & \mathrm{Spec} C \\ & \searrow h & \downarrow t & & \downarrow f \\ & & \mathcal{X} & \xrightarrow{p} & \mathrm{Spec} B \end{array}$$

Since α is a monomorphism with a section, α and s are inverses of each other. The pullback along t of the adjoint map $\delta_{\mathcal{F}}: p^*p_*\mathcal{F} \rightarrow \mathcal{F}$ is

$$t^*p^*p_*\mathcal{F} = \alpha^*(f^*p_*\mathcal{F}) \rightarrow \alpha^*\alpha_*(t^*\mathcal{F}) \rightarrow t^*\mathcal{F}$$

The first map is an isomorphism because f is flat and therefore $f^*p_*\mathcal{F} \rightarrow \alpha_*t^*\mathcal{F}$ is an isomorphism (see [Ton20, Prop 1.15]). The second map is an isomorphism because α is an isomorphism. Pulling back again along s we obtain the result.

1) \implies 3). Set $B = H^0(\mathcal{O}_{\mathcal{X}})$ and $p: \mathcal{X} \rightarrow \text{Spec } B$ the induced map. Notice that $L_B(\{\mathcal{O}_{\mathcal{X}}\}, B) \simeq \text{Mod } B$ and under this isomorphism

$$\Omega^*: \text{QCoh } \mathcal{X} \rightarrow L_B(\{\mathcal{O}_{\mathcal{X}}\}, B) \text{ and } \mathcal{F}_{*, \{\mathcal{O}_{\mathcal{X}}\}}: L_B(\{\mathcal{O}_{\mathcal{X}}\}, B) \rightarrow \text{QCoh } \mathcal{X}$$

correspond to $p_*: \text{QCoh } \mathcal{X} \rightarrow \text{Mod } B$ and $p^*: \text{Mod } B \rightarrow \text{QCoh } \mathcal{X}$ respectively. Since $\{\mathcal{O}_{\mathcal{X}}\}$ generates $\text{QCoh}(\mathcal{X})$ by hypothesis we can apply 1.6 and conclude that $p_*: \text{QCoh } \mathcal{X} \rightarrow \text{Mod } B$ is fully faithful, $p^*: \text{Mod } B \rightarrow \text{QCoh}(\mathcal{X})$ is exact, that is $p: \mathcal{X} \rightarrow \text{Spec } B$ is flat by [Ton20, Prop 1.14], and $p^*p_* \simeq \text{id}$.

We want to show that $\mathcal{X} \rightarrow \text{Spec } B$ is fully faithful or, equivalently, that the diagonal $\mathcal{X} \rightarrow \mathcal{X} \times_B \mathcal{X}$ is an equivalence. Let $h: V = \text{Spec } C \rightarrow \mathcal{X}$ be a representable fpqc covering and consider the Cartesian diagrams

$$\begin{array}{ccc} \mathcal{X}_V & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow p \\ V & \xrightarrow{f} & \text{Spec } B \end{array} \quad \begin{array}{ccc} V & \xrightarrow{h} & \mathcal{X} \\ \downarrow s & & \downarrow \Delta \\ \mathcal{X} \times_B V & \xrightarrow{\text{id} \times h} & \mathcal{X} \times_B \mathcal{X} \end{array}$$

where $f = ph: V \rightarrow \text{Spec } B$. We are going to prove that $\mathcal{X}_V \rightarrow V$ is an isomorphism, but first we show how to conclude the proof from this. Since s is a section of $\mathcal{X}_V \rightarrow V$ it would be an isomorphism too. Since the horizontal arrow in the second diagram is an fpqc covering, by descent it would follow that Δ is an equivalence. Moreover if $\mathcal{X} \times_B k \neq \emptyset$ for all geometric points then $f: V \rightarrow \text{Spec } B$ would be an fpqc covering. The fact that $\mathcal{X}_V \rightarrow V$ is an equivalence would therefore imply that $p: \mathcal{X} \rightarrow \text{Spec } B$ is an equivalence.

Thus our goal is to show that $\mathcal{X}_V \rightarrow V$ is an isomorphism. Since p is flat also f is flat. In particular $H^0(\mathcal{O}_{\mathcal{X}}) = B$ implies $H^0(\mathcal{O}_{\mathcal{X}_V}) = C$. Since $\mathcal{X}_V \rightarrow \mathcal{X}$ is affine and $\{\mathcal{O}_{\mathcal{X}}\}$ generates $\text{QCoh}(\mathcal{X})$ it follows that $\{\mathcal{O}_{\mathcal{X} \times_B V}\}$ generates $\text{QCoh}(\mathcal{X} \times_B V)$ (see [Ton20, Rem 7.3]). Thus \mathcal{X}_V is pseudo-affine.

Discussion above shows that we can assume that $p: \mathcal{X} \rightarrow \text{Spec } B$ has a section, that we still denote by $s: \text{Spec } B \rightarrow \mathcal{X}$, and we have to show that it is an isomorphism.

We first prove that $p_*: \text{QCoh } \mathcal{X} \rightarrow \text{Mod } B$ is an equivalence. It suffices to show that, if $M \in \text{Mod } B$, then the map $\gamma_M: M \rightarrow p_*p^*M$ is an isomorphism. Notice that $p^*\gamma_M$ is a section of the map $(p^*p_*)p^*M \rightarrow p^*M$ which is an isomorphism. So $p^*\gamma_M$ and $\gamma_M = s^*p^*\gamma_M$ are isomorphisms. Since $p_*s_*: \text{Mod } B \rightarrow \text{Mod } B$ is the identity, we can conclude that $s_* \simeq p^*$ and $s^* \simeq p_*$. Consider the Cartesian diagram

$$\begin{array}{ccc} U & \xrightarrow{g} & \text{Spec } C \\ \downarrow t & & \downarrow h \\ \text{Spec } B & \xrightarrow{s} & \mathcal{X} \end{array}$$

where h is a representable fpqc covering. Since \mathcal{X} has quasi-affine diagonal it follows that U is a quasi-affine scheme. Moreover

$$H^0(\mathcal{O}_U) \simeq g_*t^*\mathcal{O}_B \simeq h^*s_*\mathcal{O}_B \simeq h^*p^*\mathcal{O}_B \simeq C$$

Thus $g: U \rightarrow \text{Spec } C$ and, by descent, $s: \text{Spec } B \rightarrow \mathcal{X}$ are open immersions. We finish by proving that g is surjective. Let Z be the complement of U in $\text{Spec } C$ with reduced structure. We have

$$H^0(\mathcal{O}_Z) \simeq p_* h_* \mathcal{O}_Z \simeq s^* h_* \mathcal{O}_Z = t_* g^* \mathcal{O}_Z = 0$$

Thus Z is empty as required. \square

Remark 2.3. The assumption on the diagonal in 2.2 is necessary: the stack $\mathcal{X} = B_k E$, where E is an elliptic curve over a field k , is not a sheaf but $\text{QCoh } \mathcal{X} \simeq \text{QCoh } k$.

Remark 2.4. As a consequence of 2.2 a pseudo-affine stack is separated. In particular a pseudo-affine map $f: \mathcal{Y} \rightarrow \mathcal{X}$ of pseudo-algebraic fiber categories is quasi-compact and separated. Taking an affine atlas of \mathcal{X} and applying 2.2 we can also conclude that $f^* f_* \mathcal{F} \rightarrow \mathcal{F}$ is surjective for all $\mathcal{F} \in \text{QCoh } \mathcal{Y}$. In particular if $\mathcal{D} \subseteq \text{QCoh } \mathcal{X}$ generates $\text{QCoh } \mathcal{X}$ then $f^* \mathcal{D}$ generates $\text{QCoh } \mathcal{Y}$ (see [Ton20, Rem 7.3]).

Remark 2.5. If $\mathcal{Y} \rightarrow \text{Spec } A$ is a map of fibered categories then \mathcal{Y} is pseudo-affine if and only if $\mathcal{Y} \rightarrow \text{Spec } A$ is pseudo-affine, because if \mathcal{Y} is pseudo-affine and $g: \mathcal{Y}' \rightarrow \mathcal{Y}$ is an affine map, then \mathcal{Y}' is an fpqc sheaf with quasi-affine diagonal and $\mathcal{O}_{\mathcal{Y}'} = g^* \mathcal{O}_{\mathcal{Y}}$ generates $\text{QCoh } \mathcal{Y}'$.

Remark 2.6. Consider a Cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{X} \end{array}$$

of pseudo-algebraic fpqc stacks such that $X \rightarrow \mathcal{X}$ is a representable fpqc covering. If $Y \rightarrow X$ is pseudo-affine then so is $\mathcal{Y} \rightarrow \mathcal{X}$.

By standard arguments of descent we can assume $\mathcal{X} = \text{Spec } B$ and $X = \text{Spec } B'$ affine. So Y is pseudo-affine by hypothesis. Since $Y \rightarrow \mathcal{Y}$ is affine it follows that \mathcal{Y} is quasi-compact with affine diagonal. In particular, since $B \rightarrow B'$ is flat, we have $H^0(\mathcal{O}_{\mathcal{Y}}) \simeq H^0(\mathcal{O}_Y) \otimes_{B'} B$ and therefore we can assume $H^0(\mathcal{O}_{\mathcal{Y}}) = B$ and $H^0(\mathcal{O}_Y) = B'$. In this case $Y \rightarrow X$ is flat and fully faithful and, since \mathcal{X} and \mathcal{Y} are fpqc stacks, it follows that also $\mathcal{Y} \rightarrow \mathcal{X}$ is flat and fully faithful.

Theorem 2.7. *A quasi-compact flat monomorphism of algebraic stacks is quasi-affine.*

Proof. We have to prove that if $\mathcal{X} \rightarrow \text{Spec } R$ is a flat monomorphism and \mathcal{X} is quasi-compact algebraic space then \mathcal{X} is a quasi-affine scheme. First we observe that \mathcal{X} is separated because $\mathcal{X} \rightarrow \mathcal{X} \times_R \mathcal{X}$ is an isomorphism. Thus we can apply 2.2 and conclude that \mathcal{X} is pseudo-affine. Moreover by [Aut19, Tag 0B8A] the space \mathcal{X} is actually a scheme. By [Gro60, Prop 5.1.2] we conclude that \mathcal{X} is quasi-affine. \square

Corollary 2.8. *A pseudo-affine algebraic stack is a quasi-affine scheme.*

Proof. Use 2.2 and 2.7. \square

The following property is known for algebraic stacks (see [Gro13, Corollary 5.11] and [Tot02, Proposition 1.3]).

Corollary 2.9. *A quasi-compact fpqc stack \mathcal{X} with quasi-affine diagonal and with the resolution property has affine diagonal.*

Proof. Since \mathcal{X} is pseudo-algebraic the category $\text{Vect } \mathcal{X}$ is essentially small. Thus we can consider a set \mathcal{R} of representatives of isomorphism classes of locally free sheaves over \mathcal{X} . Given $\mathcal{E} \in \mathcal{R}$ we define the sheaf

$$\text{Fr}(\mathcal{E}) = \bigsqcup_{n \in \mathbb{N}} \underline{\text{Iso}}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}^n, \mathcal{E}): (\text{Aff}/\mathcal{X})^{\text{op}} \longrightarrow (\text{Sets}), \quad \text{Fr}(\mathcal{E})(U \xrightarrow{f} \mathcal{X}) = \bigsqcup_{n \in \mathbb{N}} \text{Iso}_U(\mathcal{O}_U^n, f^* \mathcal{E})$$

The map $\text{Fr}(\mathcal{E}) \longrightarrow \mathcal{X}$ is an affine fpqc covering: $\mathcal{X} = \sqcup_n \mathcal{X}_n$ where \mathcal{X}_n is the locus where \mathcal{E} has rank n and $\text{Fr}(\mathcal{E})|_{\mathcal{X}_n} = \underline{\text{Iso}}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}^n, \mathcal{E}) \rightarrow \mathcal{X}_n$ is a GL_n -torsor. In particular $g: \text{Fr} = \prod_{\mathcal{E} \in \mathcal{R}} \text{Fr}(\mathcal{E}) \longrightarrow \mathcal{X}$ is also an affine fpqc covering and Fr is quasi-compact with quasi-affine diagonal.

Since g is affine by 2.4 $g^* \text{Vect } \mathcal{X}$ generates $\text{QCoh } \text{Fr}$. On the other hand if $\mathcal{E} \in \text{Vect } \mathcal{X}$ then by construction Fr is a (finite) disjoint union of open substacks over which $g^* \mathcal{E}$ is free, which implies that $g^* \mathcal{E}$ is generated by global sections. We can conclude that Fr is a pseudo-affine sheaf and, by 2.4, that it has affine diagonal.

In particular, taking an affine atlas of Fr , we get an affine map faithfully flat map $V \rightarrow \mathcal{X}$ from an affine scheme. In particular $V \times_{\mathcal{X}} V$ is affine and $V \times_{\mathcal{X}} V \rightarrow V \times V$ is an affine map. Since this last map is the base change of the diagonal $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ along the fpqc covering $V \times V \rightarrow \mathcal{X} \times \mathcal{X}$ we can conclude that the diagonal $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is affine as well. \square

We prove the following result to compare our results with [Sav06].

Proposition 2.10. *Let G be a flat and affine group scheme over R such that $\text{B}_R G$ has the resolution property. Then:*

- 1) *If U is a pseudo-affine sheaf over R with an action of G then $[U/G]$ has the resolution property.*
- 2) *If $\mathcal{X} = [X/G]$ for a quasi-compact scheme X and there exists $\mathcal{L} \in \text{Pic}(X)$ whose pullback to X is very ample relatively to R then \mathcal{X} has the resolution property.*

Proof. 1) By 2.5 and 2.6 the map $[U/G] \longrightarrow \text{B}_R G$ is pseudo-affine. By 2.4 we obtain the conclusion.

2) Consider the Cartesian diagrams

$$\begin{array}{ccccc} U & \longrightarrow & \text{Spec } R & & \\ \downarrow & & \downarrow & & \\ X & \longrightarrow & \text{B}_R \mathbb{G}_m & \longrightarrow & \text{Spec } R \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \text{B}_R(G \times \mathbb{G}_m) & \longrightarrow & \text{B}_R G \end{array}$$

It follows that $\mathcal{X} \simeq [U/G \times \mathbb{G}_m]$. Now consider the Cartesian diagrams

$$\begin{array}{ccccc} U & \longrightarrow & \mathbb{A}_R^n - \{0\} & \longrightarrow & \text{Spec } R \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{i} & \mathbb{P}_R^n & \longrightarrow & \text{B}_R \mathbb{G}_m \end{array}$$

where $i: X \rightarrow \mathbb{P}_R^n$ is the immersion induced by \mathcal{L} . Since X is quasi-compact and \mathbb{P}_R^n is separated it follows that U is quasi-compact. Moreover $U \rightarrow \mathbb{A}_R^n$ is an immersion and, since \mathbb{A}_R^n is affine, it is quasi-compact. From [Aut19, Tag 01QV] we see that U is a quasi-compact open of a closed subscheme of \mathbb{A}_R^n and therefore it is quasi-affine.

Thus by 1) we just have to show that $\text{B}_R(G \times \mathbb{G}_m)$ has the resolution property. Let $a: \text{B}_R(G \times \mathbb{G}_m) = \text{B}_R G \times \text{B}_R \mathbb{G}_m \rightarrow \text{B}_R G$ be the projection and \mathcal{N} the pullback to $\text{B}_R(G \times \mathbb{G}_m)$ of the

canonical invertible sheaf of $B_R \mathbb{G}_m$. Given $\mathcal{F} \in \mathrm{QCoh}(B_R(G \times \mathbb{G}_m))$ there is a canonical map

$$\bigoplus_{n \in \mathbb{Z}} a^* a_* (\mathcal{F} \otimes \mathcal{N}^{-\otimes n}) \otimes \mathcal{N}^{\otimes n} \rightarrow \mathcal{F}$$

Going fpqc locally on $B_R G$ and using the usual representation theory of \mathbb{G}_m we see that the above map is an isomorphism. It is therefore clear that the sheaves of the form $a^* \mathcal{G} \otimes \mathcal{N}^{\otimes n}$ for $\mathcal{G} \in \mathrm{Vect}(B_R G)$ generates $\mathrm{QCoh}(B_R(G \times \mathbb{G}_m))$. \square

3. TANNAKA RECONSTRUCTION

The goal of this section is to introduce the stack of fiber functors and prove Theorem A. We will work over a base ring R and denote by A a general R -algebra.

Definition 3.1. Let \mathcal{X} be a quasi-compact fiber category over R , $\mathcal{C} \subseteq \mathrm{Vect} \mathcal{X}$ a monoidal subcategory and A an R -algebra. We define $\mathrm{SMex}_R(\mathcal{C}, A)$ as the category of contravariant, R -linear and *strong* monoidal functors $\Gamma: \mathcal{C} \rightarrow \mathrm{Mod} A$ such that, for all geometric points $\mathrm{Spec} k \rightarrow \mathrm{Spec} A$, $\Gamma \otimes_A k$ are left exact in the sense of 1.3.

Let \mathcal{Y} be another fibered category over R . A functor $\mathcal{C} \rightarrow \mathrm{Vect} \mathcal{Y}$ is said a *fiber* functor if it is a *covariant*, R -linear and strong monoidal functor which is right exact in the sense of 1.4. We denote by $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}(\mathcal{Y})$ the category of fiber functors $\mathcal{C} \rightarrow \mathrm{Vect}(\mathcal{Y})$.

We define $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ as the fiber category (not necessarily in groupoids) over R whose fiber over an R -algebra A is $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}(\mathrm{Spec} A)$ and we call $\mathcal{P}_{\mathcal{C}}$ the functor

$$(3.1) \quad \mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}, (\mathrm{Spec} A \xrightarrow{s} \mathcal{X}) \mapsto (s^*: \mathcal{C} \rightarrow \mathrm{Vect} A)$$

The fibered category $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ is called the *stack of fiber functors*.

We will prove the following result and, in particular, Theorem A.

Theorem 3.2. *Let \mathcal{X} be a quasi-compact fpqc stack over R with quasi-affine diagonal, A be an R -algebra and $\mathcal{C} \subseteq \mathrm{Vect} \mathcal{X}$ be a monoidal subcategory with duals that generates $\mathrm{QCoh} \mathcal{X}$. Then the functors*

$$\begin{array}{ccc} (\mathrm{Spec} \mathcal{A}_{\Gamma, \mathcal{C}} \rightarrow \mathcal{X}) & \longleftarrow & | (\Gamma: \mathcal{C} \rightarrow \mathrm{Mod} A) \\ \mathcal{X}(A) & \longrightarrow & \mathrm{SMex}_R(\mathcal{C}, A) \\ (s: \mathrm{Spec} A \rightarrow \mathcal{X}) & \longmapsto & ((s^*_{\mathcal{C}})^{\vee}: \mathcal{C} \rightarrow \mathrm{Vect} A) \end{array}$$

are well defined and quasi-inverses of each other. In particular the functor $\mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ is an equivalence of stacks.

An immediate corollary and generalization of Theorem 3.2 is the following.

Corollary 3.3. *Let \mathcal{X} be a quasi-compact fpqc stack over R with quasi-affine diagonal, $\mathcal{C} \subseteq \mathrm{Vect} \mathcal{X}$ be a monoidal subcategory with duals that generates $\mathrm{QCoh} \mathcal{X}$ and \mathcal{Y} be a fibered category over R . Then the functor*

$$\mathrm{Hom}(\mathcal{Y}, \mathcal{X}) \rightarrow \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}(\mathcal{Y}), (\mathcal{Y} \xrightarrow{f} \mathcal{X}) \mapsto f^*_{\mathcal{C}}: \mathcal{C} \rightarrow \mathrm{Vect}(\mathcal{Y})$$

is an equivalence of categories.

Proof. The map in the statement is obtained applying $\mathrm{Hom}(\mathcal{Y}, -)$ to the functor $\mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$, which is an equivalence by 3.2. \square

Remark 3.4. Let $R = k$ be a field, $\mathcal{X} = \Delta$ be an affine gerbe over k and $\mathcal{C} = \text{Vect}(\Delta)$, which is a k -Tannakian category. The stack of fiber functor $\Pi_{\mathcal{C}} \rightarrow \text{Aff}/k$ usually associated with \mathcal{C} is defined as

$$\Pi_{\mathcal{C}}(A) = \{k\text{-linear, exact and strong monoidal functors } \mathcal{C} \rightarrow \text{Vect}(A)\}$$

Sometimes it is also required that those functors are faithful. In any case, by [Del90, Cor 2.10] all those notions coincide and $\Pi_{\mathcal{C}} = \text{Fib}_{\Delta, \mathcal{C}}$. Classical Tannaka's reconstruction states that the functor $\mathcal{P}_{\mathcal{C}}: \Delta \rightarrow \Pi_{\mathcal{C}} = \text{Fib}_{\Delta, \mathcal{C}}$ is an equivalence. On the other hand $\mathcal{C} = \text{Vect}(\Delta)$ generates $\text{QCoh}(\Delta)$ (see [Del90, Corollary 3.9]), so that Theorem 3.2 can be seen of a generalization of classical Tannaka's reconstruction.

Remark 3.5. Even though we keep choosing a base ring R , the fibered category $\text{Fib}_{\mathcal{X}, \mathcal{C}} \rightarrow \text{Aff}/R \rightarrow \text{Aff}$ does not depend on this choice: we may have chosen $R = \mathbb{Z}$ or $R = \mathbb{H}^0(\mathcal{O}_{\mathcal{X}})$. Indeed if A is an R -algebra and $\Gamma: \mathcal{C} \rightarrow \text{Mod}(A)$ is an R -linear functor, then $\mathbb{H}^0(\mathcal{O}_{\mathcal{X}}) = \text{End}(\mathcal{O}_{\mathcal{X}}) \rightarrow \text{End}(A) = A$ makes A into a $\mathbb{H}^0(\mathcal{O}_{\mathcal{X}})$ -algebra and Γ into an $\mathbb{H}^0(\mathcal{O}_{\mathcal{X}})$ linear functor.

Lemma 3.6. *Let \mathcal{X} be a quasi-compact fibered category over R , $\mathcal{C} \subseteq \text{Vect } \mathcal{X}$ be a monoidal subcategory with duals, $\Gamma: \mathcal{C} \rightarrow \text{Vect}(A)$ be a contravariant, R -linear and strong monoidal functor and set $\mathcal{A} = \mathcal{A}_{\Gamma, \mathcal{C}}$, $f: \mathcal{Y} = \text{Spec } \mathcal{A} \rightarrow \mathcal{X}$. Then*

$$\Omega_{\mathcal{E}}^{\mathcal{A}} \simeq \mathbb{H}^0((f^* \mathcal{E})^{\vee})$$

and the map

$$\Gamma_{\mathcal{E}} \otimes_A \mathcal{O}_{\mathcal{Y}} \rightarrow (f^* \mathcal{E})^{\vee}$$

induced by $\Gamma \rightarrow \Omega^{\mathcal{A}}$ is surjective.

Proof. Set $\pi: \mathcal{X}_A \rightarrow \mathcal{X}$ for the projection. We have monoidal isomorphisms

$$\Omega_{\mathcal{E}}^{\mathcal{A}} = \text{Hom}(\pi^* \mathcal{E}, \mathcal{A}) \simeq \text{Hom}(\mathcal{E}, f_* \mathcal{O}_{\mathcal{Y}}) \simeq \text{Hom}(f^* \mathcal{E}, \mathcal{O}_{\mathcal{Y}}) = \mathbb{H}^0((f^* \mathcal{E})^{\vee})$$

In particular we obtain a monoidal natural transformation $\Gamma \rightarrow \Omega^{\mathcal{A}} = \mathbb{H}^0((f^* -)^{\vee})$ and therefore a commutative diagram

$$\begin{array}{ccc} \Gamma_{\mathcal{E}} \otimes \Gamma_{\mathcal{E}^{\vee}} & \longrightarrow & \mathbb{H}^0(f^* \mathcal{E}^{\vee}) \otimes \mathbb{H}^0(f^* \mathcal{E}) \\ \downarrow & & \downarrow \omega \\ \Gamma_{\mathcal{E} \otimes \mathcal{E}^{\vee}} & \longrightarrow & \mathbb{H}^0(f^*(\mathcal{E} \otimes \mathcal{E}^{\vee})^{\vee}) \end{array}$$

Consider the evaluation $e: \mathcal{E} \otimes \mathcal{E}^{\vee} \rightarrow \mathcal{O}_{\mathcal{X}}$, which is a map in \mathcal{C} because \mathcal{C} has duals. The morphism $\mathbb{H}^0(f^* \mathcal{O}_{\mathcal{X}}) \rightarrow \mathbb{H}^0(f^*(\mathcal{E} \otimes \mathcal{E}^{\vee})^{\vee})$ maps 1 to an element that we denote by ψ . After the usual identifications $\mathcal{E} \otimes \mathcal{E}^{\vee} \simeq \underline{\text{End}}(\mathcal{E})$, the map ω become the evaluation

$$\omega: \mathbb{H}^0(f^* \mathcal{E}) \otimes \text{Hom}_{\mathcal{Y}}(f^* \mathcal{E}, \mathcal{O}_{\mathcal{Y}}) \longrightarrow \text{End}_{\mathcal{Y}}(f^* \mathcal{E}), \quad \omega(x \otimes \phi)(y) = x\phi(y)$$

while ψ become $\text{id}_{f^* \mathcal{E}}$.

By hypothesis the vertical map on the left in the above diagram is an isomorphism. By functoriality, there exist $x_1, \dots, x_n \in \mathbb{H}^0(f^* \mathcal{E})$, $\phi_1, \dots, \phi_n \in \text{Hom}(f^* \mathcal{E}, \mathcal{O}_{\mathcal{X}})$, coming respectively from $\Gamma_{\mathcal{E}^{\vee}}$ and $\Gamma_{\mathcal{E}}$, such that $\text{id}_{f^* \mathcal{E}} = \omega(\sum_i x_i \otimes \phi_i)$. This implies that the map $\mathcal{O}_{\mathcal{Y}}^n \rightarrow f^* \mathcal{E}$ given by the global sections x_1, \dots, x_n is surjective. Since this map factors through $\Gamma_{\mathcal{E}^{\vee}} \otimes_A \mathcal{O}_{\mathcal{Y}} \rightarrow f^* \mathcal{E}$ by construction, this ends the proof. \square

Lemma 3.7. *Let \mathcal{X} be a quasi-compact fibered category over R and $\mathcal{C} \subseteq \text{Vect } \mathcal{X}$ be a monoidal subcategory. If $\Gamma \in \text{SMex}_R(\mathcal{C}, A)$ and $\mathcal{E}' \xrightarrow{\beta} \mathcal{E} \rightarrow 0$ or $\mathcal{E}'' \xrightarrow{\alpha} \mathcal{E}' \xrightarrow{\beta} \mathcal{E} \rightarrow 0$ is a finite test sequence in \mathcal{C} then*

$$0 \rightarrow \Gamma_{\mathcal{E}} \rightarrow \Gamma_{\mathcal{E}'} \rightarrow \text{Coker}(\Gamma_{\beta}) \rightarrow 0 \text{ and } 0 \rightarrow \text{Coker}(\Gamma_{\beta}) \rightarrow \Gamma_{\mathcal{E}''} \rightarrow \text{Coker}(\Gamma_{\alpha}) \rightarrow 0$$

are short exact sequences of vector bundles over A .

In particular Γ , as well as any base change $\Gamma \otimes_A B$ for an A -algebra B , are left exact. Moreover $\Gamma \otimes_A B \in \text{SMex}_R(\mathcal{C}, B)$.

Proof. By [Aut19, Tag 046Y] if $M \rightarrow N$ is a map of vector bundles over A and, for all $p \in \text{Spec } A$, the map $M \otimes_A k(p) \rightarrow N \otimes_A k(p)$ is injective then $M \rightarrow N$ is injective and N/M is flat. As N/M is finitely presented we can moreover conclude that N/M is a vector bundle. Applying this on the map $\Gamma_{\mathcal{E}} \rightarrow \Gamma_{\mathcal{E}'}$ we obtain the first exact sequence and that $\text{Coker}(\Gamma_{\beta})$ is locally free. Here we use the definition of SMex , which also tell us that $\gamma: \text{Coker}(\Gamma_{\beta}) \rightarrow \Gamma_{\mathcal{E}''}$ is injective on the geometric point. Again we can conclude that this map is injective and that $\text{Coker}(\gamma)$ is a vector bundle. As $\text{Coker}(\gamma) = \text{Coker}(\Gamma_{\alpha})$ this concludes the proof. \square

By definition $\text{SMex}_R(\mathcal{C}, -)$ is a stack (not necessarily in groupoids) over Aff/R . The next result tells us that this stack is just $\text{Fib}_{\mathcal{X}, \mathcal{C}}$.

Proposition 3.8. *Let \mathcal{X} be a quasi-compact fibered category over R and $\mathcal{C} \subseteq \text{Vect } \mathcal{X}$ be a monoidal subcategory. Then*

$$\text{Fib}_{\mathcal{X}, \mathcal{C}} \rightarrow \text{SMex}_R(\mathcal{C}, -), \quad \Gamma \mapsto \Gamma^{\vee}$$

is an equivalence of stacks over Aff/R .

Proof. If $\Gamma: \mathcal{C} \rightarrow \text{Vect}(A)$ is a covariant (resp. contravariant), R -linear and strong monoidal functor then $\Gamma^{\vee}: \mathcal{C} \rightarrow \text{Vect}(A)$ is a contravariant (resp. covariant) R -linear functor which moreover has a strong monoidal structure. Moreover if B is an A -algebra we have

$$\text{Hom}_A(\Gamma_{\mathcal{E}}, A) \otimes_A B \simeq \text{Hom}_B(\Gamma_{\mathcal{E}} \otimes_A B, B)$$

because $\Gamma_{\mathcal{E}} \in \text{Vect}(A)$.

If $\Gamma \in \text{SMex}_R(\mathcal{C}, A)$ then $\Gamma^{\vee} \in \text{Fib}_{\mathcal{X}, \mathcal{C}}(A)$ thanks to 3.7. Conversely assume $\Gamma \in \text{Fib}_{\mathcal{X}, \mathcal{C}}(A)$. We must show that $(\Gamma \otimes_A k)^{\vee} \simeq \Gamma^{\vee} \otimes_A k$ is left exact for all geometric points $\text{Spec } k \rightarrow \text{Spec } A$. As $\Gamma \otimes_A k \in \text{Fib}_{\mathcal{X}, \mathcal{C}}(k)$ we can assume that A is a field. The functor $\Gamma^{\vee}: \mathcal{C} \rightarrow \text{Vect}(A)$ is left exact because Γ is exact on all finite test sequences and the dual of a right exact sequence is again exact. \square

Proof. (of Theorem 3.2). The last claim follows from 3.8. Set $\pi: \mathcal{X}_A \rightarrow \mathcal{X}$ for the projection. Composing by π we obtain an equivalence $\mathcal{X}_A(A) \rightarrow \mathcal{X}(A)$. Since \mathcal{X} and therefore \mathcal{X}_A have affine diagonal by 2.9, all morphisms $\text{Spec } A \rightarrow \mathcal{X}, \mathcal{X}_A$ are affine. Therefore the functor $\mathcal{X}_A(A) \rightarrow \text{QAlg}(\mathcal{X}_A)$ which maps $t: \text{Spec } A \rightarrow \mathcal{X}_A$ to $t_* \mathcal{O}_A$ is fully faithful. By 1.6 and the fact that

$$\Omega_{\mathcal{E}}^{t_* \mathcal{O}_A} = \text{Hom}_{\mathcal{X}_A}(\pi^* \mathcal{E}, t_* \mathcal{O}_A) \simeq \text{Hom}_{\mathcal{X}}(\mathcal{E}, s_* \mathcal{O}_A) \simeq (s^* \mathcal{E})^{\vee} \text{ for } t \in \mathcal{X}_A(A), \mathcal{E} \in \mathcal{C}$$

where $s = \pi t: \text{Spec } A \rightarrow \mathcal{X}$ we can conclude that the functor $\mathcal{X}(A) \rightarrow \text{SMex}_R(\mathcal{C}, A)$, $s \mapsto (s^*|_{\mathcal{C}})^{\vee}$ is well defined and fully faithful.

Set $\mathcal{A} = \mathcal{A}_{\Gamma, \mathcal{C}} \in \text{QAlg}(\mathcal{X}_A)$. We must show that, given $\Gamma \in \text{SMex}_R(\mathcal{C}, A)$, the composition $p: \text{Spec } \mathcal{A} \rightarrow \mathcal{X}_A \rightarrow \text{Spec } A$ is an isomorphism. Set $\mathcal{Y} = \text{Spec } \mathcal{A}$ and $f: \mathcal{Y} \rightarrow \mathcal{X}$ for the structure morphism. We want to apply 2.2 on $p: \mathcal{Y} \rightarrow \text{Spec } A$.

Notice that, since $f: \mathcal{Y} \rightarrow \mathcal{X}$ is affine, \mathcal{Y} is quasi-compact and has affine diagonal. Moreover by 1.6 and 3.6 we have monoidal isomorphisms

$$\Gamma_{\mathcal{E}} \simeq \Omega_{\mathcal{E}}^{\mathcal{A}} \simeq \text{H}^0((f^* \mathcal{E})^{\vee})$$

In particular, since Γ is a strong monoidal functor, the isomorphism $A \rightarrow \Gamma_{\mathcal{O}_{\mathcal{X}}}$ yields the isomorphism $A \rightarrow \text{H}^0(\mathcal{A})$.

Let $\text{Spec } k \rightarrow \text{Spec } A$ be a geometric point and set $g: \mathcal{X}_k \rightarrow \mathcal{X}_A$ for the base change map. We have $\mathcal{Y} \times_A k \simeq \text{Spec}(g^* \mathcal{A})$, while by [Ton20, Prop 2.14] we have $g^* \mathcal{A} \simeq \mathcal{A}_{\Gamma \otimes_A k, \mathcal{C}}$. Since $\Gamma \otimes_A k$

is also left exact by hypothesis, by 1.6 we get $\Gamma \otimes_A k \simeq \Omega^{\mathcal{A}_{\Gamma \otimes_A k, C}}$. Thus $\Gamma_{\mathcal{O}_{\mathcal{X}}} \otimes_A k = A \otimes_A k \simeq k$ implies that $\Omega_{\mathcal{O}_{\mathcal{X}}}^{\mathcal{A}_{\Gamma \otimes_A k, C}} \simeq k$ and therefore $\mathcal{A}_{\Gamma \otimes_A k, C} \neq 0$, that is $\mathcal{Y} \times_A k \neq \emptyset$.

It remains to show that $\{\mathcal{O}_{\mathcal{Y}}\}$ generates $\text{QCoh } \mathcal{Y}$. Since $\mathcal{Y} \xrightarrow{f} \mathcal{X}$ is affine, $f^* \mathcal{C}$ generates $\text{QCoh } \mathcal{Y}$ by 2.4. On the other hand every sheaf $f^* \mathcal{E}$ is generated by global sections thanks to 3.6. In conclusion \mathcal{Y} is pseudo-affine and, by 2.2, the map $p: \mathcal{Y} \rightarrow \text{Spec } A$ is an isomorphism. \square

4. PSEUDO-AFFINE SHEAVES REVISITED

In this section we give an alternative characterization of pseudo-affine sheaves, which will be used when studying the fiber category $\text{Fib}_{\mathcal{X}, C}$.

Lemma 4.1. *Let $\pi: \mathcal{X} \rightarrow \text{Aff}/R$ be a fibered category over R , \mathcal{T}_* be a bounded above complex of locally free sheaves on \mathcal{X} and denote by \mathcal{U}_l , for $l \in \mathbb{Z}$, the locus in \mathcal{X} where \mathcal{T}_* is exact in degrees greater than l , that is*

$$\mathcal{U}_l(A) = \{\xi: \text{Spec } A \rightarrow \mathcal{X} \mid \xi^* \mathcal{T}_* \text{ is exact in degrees greater than } l\} \subseteq \mathcal{X}(A)$$

Then $\mathcal{U}_l \rightarrow \mathcal{X}$ is a quasi-compact open immersion.

Proof. Since \mathcal{T}_* is bounded above we have $\mathcal{U}_l = \mathcal{X}$ for $l \gg 0$. It is therefore enough to show that $\mathcal{U}_{l-1} \rightarrow \mathcal{U}_l$ is a quasi-compact open immersion. In particular we can assume \mathcal{X} affine and $\mathcal{U}_l = \mathcal{X}$, that is assume \mathcal{T}_* exact in degrees greater than l . Set $\mathcal{E} = \text{Ker}(\mathcal{T}_l \rightarrow \mathcal{T}_{l+1})$ and consider the complex

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{T}_l \rightarrow \mathcal{T}_{l+1} \rightarrow \cdots$$

By construction this complex is exact. Since all the \mathcal{T}_l are locally free it follows that \mathcal{E} is locally free as well and that this complex remains exact after any pullback. We can therefore conclude that \mathcal{U}_{l-1} is the locus where the map

$$\mathcal{T}_{l-1} \rightarrow \text{Ker}(\mathcal{T}_l \rightarrow \mathcal{T}_{l+1}) = \mathcal{E}$$

is surjective. In particular \mathcal{U}_{l-1} is a subfibered category of \mathcal{X} . Going Zariski locally we can assume \mathcal{T}_{l-1} and \mathcal{E} free of rank m and n . In this case the locus \mathcal{U}_l is the locus where

$$\Lambda^n \mathcal{T}_{l-1} \rightarrow \Lambda^n \mathcal{E} = \det \mathcal{E} \simeq B$$

is surjective. But this is the complement of the zero locus defined by the above matrix, which is a quasi-compact open subset of $\text{Spec } B$. \square

Theorem 4.2. *An intersection of quasi-compact open subschemes (thought of as sheaves) of an affine scheme is pseudo-affine. Conversely if U is a pseudo-affine stack then it is (equivalent to) a sheaf and it is the intersection of the quasi-compact open subschemes of $\text{Spec } H^0(\mathcal{O}_U)$ containing it.*

Proof. Let $X = \text{Spec } B$ be an affine scheme, $\{U_i\}_{i \in I}$ be a set of quasi-compact open subsets of X and set $U = \bigcap_i U_i$. If $i \in I$ the subscheme U_i is the complement of the zero locus of finitely many elements of B and thus there exists a free B -module E_i and a map $\phi_i: E_i \rightarrow B$ such that U_i is the locus where ϕ_i is surjective. Let $V_i: \text{Aff}/B \rightarrow (\text{Sets})$ be the functor

$$V_i(A) = \{s \in E_i \otimes_B A \mid \phi_i(s) = 1\}$$

It is easy to check that V_i is affine, the map $V_i \rightarrow \text{Spec } B$ factors through U_i and that $V_i \rightarrow U_i$ is locally $(\text{Ker } \phi_i)|_{U_i} \rightarrow U_i$. In particular $V_i \rightarrow \text{Spec } B$ is flat. Now set $V = \prod_i V_i: \text{Aff}/B \rightarrow (\text{Sets})$, that is

$$V(A) = \{(s_i)_{i \in I} \mid s_i \in E_i \otimes_B A \text{ and } \phi_i(s_i) = 1\}$$

The scheme V is affine, the map $V \rightarrow \text{Spec } B$ is flat, factors through U and $V \rightarrow U$ is surjective (as functors). Moreover if $\text{Spec } A \rightarrow U$ is any map then $V \times_U A = V \times_X A$ over A because

$U \rightarrow X = \text{Spec } B$ is a monomorphism. It follows that $V \rightarrow U$ is affine and faithfully flat, in particular an fpqc covering. Thus U is quasi-compact and $U \rightarrow X$ is a flat monomorphism. The result then follows from 2.2.

Now assume that U is a pseudo-affine stack. By 2.2 we know that $p: U \rightarrow \text{Spec } B$, $B = H^0(\mathcal{O}_U)$, is a flat monomorphism. Denote by Z the intersection of all quasi-compact open subsets of $\text{Spec } B$ containing U and set $\mathcal{C} = \{\mathcal{O}_U\}$. In particular $U \subseteq Z$.

Given a map $\alpha: \text{Spec } A \rightarrow \text{Spec } B$ factoring through Z we have to show that it factors through U . Consider $\alpha|_{\mathcal{C}}: \mathcal{C} \rightarrow \text{Vect}(A)$, which is a covariant strong monoidal functor. We show that $\alpha|_{\mathcal{C}} \in \text{Fib}_{U, \mathcal{C}}(A)$. By definition, if $\mathcal{T}_*: \mathcal{O}_U^m \rightarrow \mathcal{O}_U^q \rightarrow 0$ or $\mathcal{T}_*: \mathcal{O}_U^n \rightarrow \mathcal{O}_U^m \rightarrow \mathcal{O}_U^q \rightarrow 0$ is an exact sequence on U , we must show that $\alpha|_{\mathcal{C}}$ is exact on \mathcal{T}_* . The sequence \mathcal{T}_* defines a complex \mathcal{W}_* of free A -modules, namely $\mathcal{W}_* = H^0(\mathcal{T}_*)$, and the locus W in $\text{Spec } A$ where \mathcal{W}_* is exact is quasi-compact, open and contains U (see 4.1). Thus $Z \subseteq W$, the sequence \mathcal{W}_* become exact on Z and therefore α^* maintains its exactness, as required.

Since U is pseudo-affine and $\alpha|_{\mathcal{C}} \in \text{Fib}_{U, \mathcal{C}}(A)$, by 3.2 there exists $s: \text{Spec } A \rightarrow U$ and an isomorphism $s|_{\mathcal{C}} \simeq \alpha|_{\mathcal{C}}$ in $\text{Fib}_{U, \mathcal{C}}(A)$. On the other hand, since $B = H^0(\mathcal{O}_U)$, the restriction $\{\mathcal{O}_{\text{Spec } B}\} \rightarrow \mathcal{C} = \{\mathcal{O}_U\}$ is an equivalence and therefore $\text{Fib}_{U, \mathcal{C}}(A) \rightarrow \text{Fib}_{\text{Spec } B, \{\mathcal{O}_{\text{Spec } B}\}}$ is fully faithful. By 3.2 applied on $\text{Spec } B$ we can conclude that $\alpha: \text{Spec } A \rightarrow \text{Spec } B$ and $\text{Spec } A \xrightarrow{s} U \rightarrow \text{Spec } B$ coincides as required. \square

Example 4.3. We show an example of a pseudo-affine sheaf which is not quasi-affine. Let B be a noetherian normal domain with $\dim B \geq 2$ and infinitely many primes p with $\text{ht } p = 2$. For instance $B = k[x, y]$ for a field k . Let $I = \{p \in \text{Spec } B \mid \text{ht } p \geq 2\}$ and $U_p = \text{Spec } B - V(p)$ for $p \in I$. Set $U = \bigcap_{p \in I} U_p$. The sheaf U is pseudo-affine by 4.2 and we are going to show that it is not quasi-affine. Assume by contradiction that this is true. We identify the points of the topological space of U with the ones of its image in $\text{Spec } B$. Remember that a map $\text{Spec } C \rightarrow \text{Spec } B$ factors through U if and only if $pC = C$ for all $p \in I$. Given $q \in \text{Spec } B$ and $k(q) \subseteq L$ an extension of fields, we have that $\text{Spec } L \rightarrow \text{Spec } B$ factors through U if and only if

$$pL = L \forall p \in I \iff pk(q) = k(q) \forall p \in I \iff pB_q = B_q \forall p \in I \iff \text{ht } q \leq 1$$

This tells us that $U = \{q \in \text{Spec } B \mid \text{ht } q \leq 1\}$ and that, for all $q \in U$, $\text{Spec } B_q$ factors through U . So, if $q \in U$, The map $U \times_B B_q \rightarrow \text{Spec } B_q$ is a monomorphism with a section and therefore an isomorphism. In particular $\mathcal{O}_{U, q} = B_q$. If $V = \text{Spec } A \subseteq U$ is an open subset and $q \in \text{Spec } A$ is a minimal prime, then $A_q = B_q$ has dimension 0 and therefore q must be the generic point of $\text{Spec } B$. Thus U is irreducible and, since the local rings of U are all domains, integral. In particular

$$H^0(\mathcal{O}_U) = \bigcap_{q \in U} \mathcal{O}_{U, q} = \bigcap_{q \in \text{Spec } B \mid \text{ht } q \leq 1} B_q = B$$

where the last equality follows from the fact that B is normal. Since U is quasi-affine, it follows that it is an open subset of $\text{Spec } B$. So $Z = \text{Spec } B - U$ has a finite number of generic points and all $p \in \text{Spec } B$ with $\text{ht } p = 2$ have to be generic points, contradicting our assumptions.

5. THE STACK OF FIBER FUNCTORS

The last statement of Theorem 3.2 admits an almost converse. Let \mathcal{X} be a fibered category over R and $\mathcal{C} \subseteq \text{Vect } \mathcal{X}$ be a full monoidal subcategory. We define

$$\mathcal{G}_*: \mathcal{C} \rightarrow \text{Hom}(\text{Fib}_{\mathcal{X}, \mathcal{C}}, \text{Vect}_R)$$

where Vect_R is the stack of locally free sheaves over R , mapping a sheaf $\mathcal{E} \in \mathcal{C}$ to

$$\mathcal{G}_{\mathcal{E}}(\Gamma \in \text{Fib}_{\mathcal{X}, \mathcal{C}}(A)) = \Gamma_{\mathcal{E}} \in \text{Vect}(A) \text{ for all } R\text{-algebras } A$$

In particular the composition $\mathcal{X} \rightarrow \text{Fib}_{\mathcal{X},\mathcal{C}} \xrightarrow{\mathcal{G}_\varepsilon} \text{Vect}_R$ is just $\mathcal{E} \in \mathcal{C}$. Notice that, a priori, $\text{Fib}_{\mathcal{X},\mathcal{C}}$ is not necessarily fibered in groupoids and therefore the notion of a locally free sheaf on it is not defined (although one can easily guess the definition). In the next result we will see that, under suitable conditions on \mathcal{C} , the stack $\text{Fib}_{\mathcal{X},\mathcal{C}}$ is fibered in groupoid. In this case \mathcal{G}_* will just be a map $\mathcal{G}_* : \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}})$, which is easily seen to be a covariant strong monoidal functor.

Given a function $f : \mathcal{C} \rightarrow \mathbb{N}$ we define $\text{Fib}_{\mathcal{X},\mathcal{C}}^f$ as the sub-fibered category of $\text{Fib}_{\mathcal{X},\mathcal{C}}$ of functors Γ such that $\text{rk} \Gamma_\varepsilon = f(\varepsilon)$ for all $\varepsilon \in \mathcal{C}$.

We summarize all the main results of this section in the following:

Theorem 5.1. *Let \mathcal{X} be a quasi-compact fibered category over R and $\mathcal{C} \subseteq \text{Vect } \mathcal{X}$ be a full monoidal subcategory with duals. If R is not a \mathbb{Q} -algebra assume moreover that $\text{Sym}^n \mathcal{E} \in \mathcal{C}$ for all $\mathcal{E} \in \mathcal{C}$ and $n \in \mathbb{N}$. Set also*

$$\mathcal{I} = \{f : \mathcal{C} \rightarrow \mathbb{N} \mid \exists \xi : \text{Spec } L \rightarrow \mathcal{X} \text{ with } f(\varepsilon) = \text{rk}(\xi^* \mathcal{E}) \text{ for all } \varepsilon \in \mathcal{C}\}$$

Then $\text{Fib}_{\mathcal{X},\mathcal{C}}$ is an fpqc stack in groupoids and:

- 1) $\text{Fib}_{\mathcal{X},\mathcal{C}}^f \neq \emptyset$ if and only if $f \in \mathcal{I}$.
- 2) Using notation from Section 1.2, we have $D(\varepsilon) = D(\mathcal{G}_\varepsilon)$ and $Q(\varepsilon) = Q(\mathcal{G}_\varepsilon)$ for $\varepsilon \in \mathcal{C}$, so that $D(\mathcal{C}) = D(\mathcal{C} \rightarrow \text{Vect}(\mathcal{X})) = D(\mathcal{G}_* : \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}}))$ is an affine scheme. If we think $\text{GL}(Q(\varepsilon))$ as a group scheme over $D(\mathcal{C})$ and set $G = \prod_{\varepsilon} \text{GL}(\varepsilon) \rightarrow D(\mathcal{C})$ then the map (see 1.13)

$$\text{Fib}_{\mathcal{X},\mathcal{C}} \rightarrow \text{B}G$$

is pseudo-affine. In particular $U = \text{Fr}(\mathcal{G}_* : \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}}))$ is pseudo-affine and $\text{Fib}_{\mathcal{X},\mathcal{C}} \simeq [U/G]$.

- 3) $\text{Fib}_{\mathcal{X},\mathcal{C}}$ is a quasi-compact fpqc stack with affine diagonal and the subcategory $\{\mathcal{G}_\varepsilon\}_{\varepsilon \in \mathcal{C}} \subseteq \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}})$ generates $\text{QCoh}(\text{Fib}_{\mathcal{X},\mathcal{C}})$. In particular $\text{Fib}_{\mathcal{X},\mathcal{C}}$ has the resolution property.
- 4) If $f \in \mathcal{I}$ then the map $\text{Spec } R \rightarrow D(\mathcal{C})$ induced by the constant function $(f(\varepsilon))_\varepsilon : \text{Spec } R \rightarrow \prod_{\varepsilon} \text{ranks}(\varepsilon)$ (see 1.9) is well defined and we have Cartesian diagrams

$$\begin{array}{ccccc} \text{Fib}_{\mathcal{X},\mathcal{C}}^f & \xrightarrow{\omega} & \text{BGL}_f & \longrightarrow & \text{Spec } R \\ \downarrow & & \downarrow & & \downarrow \\ \text{Fib}_{\mathcal{X},\mathcal{C}} & \longrightarrow & \text{B}G & \longrightarrow & D(\mathcal{C}) \end{array}$$

where $\text{GL}_f = \prod_{\varepsilon} \text{GL}_{f(\varepsilon)}$ and ω is induced by the $(\mathcal{G}_\varepsilon)_{\varepsilon \in \mathcal{C}}$. The vertical maps are flat closed immersion, $\omega : \text{Fib}_{\mathcal{X},\mathcal{C}}^f \rightarrow \text{BGL}_f$ is pseudo-affine, $U = \text{Fr}(\mathcal{G}_* : \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}}^f))$ is pseudo-affine, $\text{Fib}_{\mathcal{X},\mathcal{C}}^f \simeq [U/\text{GL}_f]$ and $\{\mathcal{G}_\varepsilon\}_{\varepsilon \in \mathcal{C}} \subseteq \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}}^f)$ generates $\text{QCoh}(\text{Fib}_{\mathcal{X},\mathcal{C}}^f)$. In particular $\text{Fib}_{\mathcal{X},\mathcal{C}}^f$ is a quasi-compact fpqc stack in groupoids with affine diagonal and the resolution property.

- 5) Assume the category \mathcal{C} has the following two properties: there exists a finite set J of objects of \mathcal{C} such that any objects of \mathcal{C} can be obtained by sheaves of J using the operations $\otimes, \oplus, -^\vee, \text{Sym}^m, \Lambda^m$ several times; there is a finite set of finite test sequences for \mathcal{C} such that if $\Gamma : \mathcal{C} \rightarrow \text{Vect}(A)$ is an R -linear strong monoidal functor with then Γ is exact. Then $\text{Fib}_{\mathcal{X},\mathcal{C}}$ is an algebraic stack and a quotient $[U'/\text{GL}_m]$ for some quasi-affine scheme U' .

Remark 5.2. It is not clear if $\text{Fib}_{\mathcal{X},\mathcal{C}}^f$ is algebraic even if \mathcal{X} is a projective scheme and we choose a \mathcal{C} that does not generate $\text{QCoh}(\mathcal{X})$. The statement of 5.1, 5) is not really satisfactory for two reasons. The first condition requires \mathcal{C} to be not too big, for example $\mathcal{C} = \text{Vect}(\mathcal{X})$ in general will not have this property. The second problem is that it is not clear when the second condition is satisfied.

5.1. Proof of Theorem 5.1. The entire section is dedicated to the proof of Theorem 5.1. In particular we assume the hypothesis and notation in its statement.

It is easy to see that $\text{Fib}_{\mathcal{X},\mathcal{C}}$ is a stack (not necessarily in groupoids) for the fpqc topology on Aff/R . To avoid problems with disjoint unions we can assume that \mathcal{X} is a Zariski stack.

Given a finite subset J of \mathcal{C} denote by \mathcal{I}_J the set of $f: J \rightarrow \mathbb{N}$ extending to a function of \mathcal{I} . In other words a function $f: J \rightarrow \mathbb{N}$ belongs to \mathcal{I}_J if and only if there exists a point $s: \text{Spec } L \rightarrow \mathcal{X}$ such that $f(\mathcal{E}) = \text{rk } s^*\mathcal{E}$ for all $\mathcal{E} \in J$. Given $f: J \rightarrow \mathbb{N}$ we denote by \mathcal{X}_f the open locus of \mathcal{X} where $\text{rk } \mathcal{E} = f(\mathcal{E})$ for all $\mathcal{E} \in J$ and set $\text{GL}_f = \prod_{\mathcal{E} \in J} \text{GL}_{f(\mathcal{E})}$. Notice that $\mathcal{X} = \bigsqcup_{f \in \mathcal{I}_J} \mathcal{X}_f$ and, since \mathcal{X} is quasi-compact, \mathcal{I}_J is finite. The sheaves $(\mathcal{E}|_{\mathcal{X}_f})_{\mathcal{E} \in J}$ induces a map $\mathcal{X}_f \rightarrow \text{BGL}_f = \mathcal{B}_f$ and thus a map

$$\omega_J: \mathcal{X} \longrightarrow \bigsqcup_{f \in \mathcal{I}_J} \mathcal{B}_f = \mathcal{B}_J$$

For all $\mathcal{E} \in J$ there is a locally free sheaf $\mathcal{H}_{\mathcal{E},J}$ on \mathcal{B}_J such that $(\mathcal{H}_{\mathcal{E},J})|_{\mathcal{B}_f}$ is the canonical locally free sheaf of rank $f(\mathcal{E})$ pullback from $\text{BGL}_{f(\mathcal{E})}$. By construction $\omega_J^*\mathcal{H}_{\mathcal{E},J} \simeq \mathcal{E}$. Set \mathcal{D}_J for the subcategory of $\text{Vect}(\mathcal{B}_J)$ consisting of all sheaves \mathcal{G} such that $\omega_J^*\mathcal{G} \in \mathcal{C}$. This is a monoidal subcategory with duals. Finally when $J = \{\mathcal{E}\}$ we will replace J by \mathcal{E} in the subscripts.

The idea now is to prove that \mathcal{D}_J generates $\text{QCoh}(\mathcal{B}_J)$, so that $\mathcal{P}_{\mathcal{D}_J}: \mathcal{B}_J \rightarrow \text{Fib}_{\mathcal{B}_J, \mathcal{D}_J}$ is an equivalence of stacks by 3.2. Using this, any time we will have a monoidal functor $\Gamma: \mathcal{C} \rightarrow \text{Vect}(A)$ in $\text{Fib}_{\mathcal{X},\mathcal{C}}(A)$, we can compose it via the pullback $\omega_J^*: \mathcal{D}_J \rightarrow \mathcal{C}$ and deduce that $\Gamma \circ \omega_J^* \simeq s^*$ for some $s: \text{Spec } A \rightarrow \mathcal{B}_J$.

Lemma 5.3. *The category \mathcal{D}_J generates $\text{QCoh}(\mathcal{B}_J)$ and, in particular, $\mathcal{P}_{\mathcal{D}_J}: \mathcal{B}_J \rightarrow \text{Fib}_{\mathcal{B}_J, \mathcal{D}_J}$ is an equivalence.*

Proof. By hypothesis \mathcal{D}_J contains $\mathcal{H}_{\mathcal{E},J}$ and therefore $\mathcal{H}_{\mathcal{E},J}^\vee$ for $\mathcal{E} \in J$. Moreover if R is not a \mathbb{Q} -algebra then \mathcal{D}_J contains also $\text{Sym}^n \mathcal{H}_{\mathcal{E},J}$ and therefore $(\text{Sym}^n \mathcal{H}_{\mathcal{E},J})^\vee$ for $n \in \mathbb{N}$ and $\mathcal{E} \in J$. So everything follows from 1.16 and 3.2. \square

Lemma 5.4. *The stack $\text{Fib}_{\mathcal{X},\mathcal{C}}$ is fibered in groupoids.*

Proof. If $\Gamma, \Gamma' \in \text{Fib}_{\mathcal{X},\mathcal{C}}(A)$, $\delta: \Gamma \rightarrow \Gamma'$ is a morphism and $\mathcal{E} \in \mathcal{C}$ then $\delta_{\mathcal{E}}: \Gamma_{\mathcal{E}} \rightarrow \Gamma'_{\mathcal{E}}$ is an isomorphism because $\Gamma \circ \omega_{\mathcal{E}}^* \xrightarrow{\delta \circ \omega_{\mathcal{E}}^*} \Gamma' \circ \omega_{\mathcal{E}}^*$ is a morphism in the groupoid $\text{Fib}_{\mathcal{B}_{\mathcal{E}}, \mathcal{D}_{\mathcal{E}}}(A) \simeq \mathcal{B}_{\mathcal{E}}(A)$. \square

Lemma 5.5. *Let t be any of the following operations of sheaves $t = -^\vee, \text{Sym}^m, \Lambda^m$ and $\mathcal{E} \in \mathcal{C}$ be such that $t(\mathcal{E}) \in \mathcal{C}$. Then $\mathcal{G}_{t(\mathcal{E})} \simeq t(\mathcal{G}_{\mathcal{E}})$ on $\text{Fib}_{\mathcal{X},\mathcal{C}}$. In particular for any $\Gamma \in \text{Fib}_{\mathcal{X},\mathcal{C}}(A)$ we get an isomorphism $\Gamma_{t(\mathcal{E})} \simeq t(\Gamma_{\mathcal{E}})$ by testing the previous isomorphism in Γ .*

Proof. Consider $J = \{\mathcal{E}\} \subseteq \mathcal{C}$. The functor $\omega_{\mathcal{E}}: \mathcal{X} \rightarrow \mathcal{B}_{\mathcal{E}}$ induces $\omega_{\mathcal{E}}^*: \mathcal{D}_{\mathcal{E}} \rightarrow \mathcal{C}$ and a functor $\delta: \text{Fib}_{\mathcal{X},\mathcal{C}} \rightarrow \text{Fib}_{\mathcal{B}_{\mathcal{E}}, \mathcal{D}_{\mathcal{E}}} \simeq \mathcal{B}_{\mathcal{E}}$. In other words there is a canonical isomorphism

$$\delta(\Gamma)|_{\mathcal{D}_{\mathcal{E}}}^* \simeq \Gamma \circ \omega_{\mathcal{E}}^*: \mathcal{D}_{\mathcal{E}} \rightarrow \text{Vect } A \text{ for } \Gamma \in \text{Fib}_{\mathcal{X},\mathcal{C}}(A)$$

Since $t(\mathcal{E}) \simeq \omega_{\mathcal{E}}^*t(\mathcal{H}_{\mathcal{E},\{\mathcal{E}\}}) \in \mathcal{C}$ we have $t(\mathcal{H}_{\mathcal{E},\{\mathcal{E}\}}) \in \mathcal{D}_{\mathcal{E}}$. So we have isomorphisms

$$\mathcal{G}_{t(\mathcal{E})}(\Gamma) = \Gamma_{t(\mathcal{E})} \simeq (\Gamma \circ \omega_{\mathcal{E}}^*)_{t(\mathcal{H}_{\mathcal{E},\{\mathcal{E}\}})} \simeq \delta(\Gamma)^*t(\mathcal{H}_{\mathcal{E},\{\mathcal{E}\}}) \simeq t(\delta(\Gamma)^*\mathcal{H}_{\mathcal{E},\{\mathcal{E}\}}) \simeq t(\Gamma_{\mathcal{E}}) \simeq t(\mathcal{G}_{\mathcal{E}})(\Gamma)$$

natural in $\Gamma \in \text{Fib}_{\mathcal{X},\mathcal{C}}$ and thus that $\mathcal{G}_{t(\mathcal{E})} \simeq t(\mathcal{G}_{\mathcal{E}})$. \square

Proposition 5.6. *We have $\text{Fib}_{\mathcal{X},\mathcal{C}}^f \neq \emptyset$ if and only if $f \in \mathcal{I}$.*

Proof. For the if part, if $f \in \mathcal{I}$ there exists $s: \text{Spec } L \rightarrow \mathcal{X}$ such that $\text{rk } s^*\mathcal{E} = f(\mathcal{E})$ for all $\mathcal{E} \in \mathcal{C}$. Thus $s|_{\mathcal{C}}^f \in \text{Fib}_{\mathcal{X},\mathcal{C}}^f(L)$. Conversely we must show that, if L is an algebraically closed field and $\Gamma \in \text{Fib}_{\mathcal{X},\mathcal{C}}(L)$, then $\text{rk } \Gamma_*: \mathcal{C} \rightarrow \mathbb{N}$ belongs to \mathcal{I} .

Given a finite subset $J \subseteq \mathcal{C}$ consider $[\Gamma \circ \omega_J^*: \mathcal{D}_J \rightarrow \text{Vect}(L)] \in \text{Fib}_{\mathcal{B}_J, \mathcal{D}_J}(L) \simeq \mathcal{B}_J(L)$, so that there exists a map $\text{Spec } L \xrightarrow{s} \mathcal{B}_J$ such that $\Gamma \circ \omega_J^* \simeq s|_{\mathcal{D}_J}^*$. The map s has image in some component \mathcal{B}_f with $f \in \mathcal{I}_J$. In particular if $\mathcal{E} \in J$ we have

$$\text{rk } \Gamma_{\mathcal{E}} = \text{rk } \Gamma_{\omega_J^* \mathcal{H}_{\mathcal{E}, J}} = \text{rk}(s^*(\mathcal{H}_{\mathcal{E}, J})|_{\mathcal{B}_f}) = f(\mathcal{E})$$

This shows that for all finite subsets $J \subseteq \mathcal{C}$ we have that $f_J = (\text{rk } \Gamma_*)|_J$ belongs to \mathcal{I}_J . By construction \mathcal{X}_{f_J} is a non-empty (open and) closed substack of \mathcal{X} . Since $\mathcal{X}_{f_J} \subseteq \mathcal{X}_{f_{J'}}$ if $J' \subseteq J$ and \mathcal{X} is quasi-compact it follows that $\bigcap_J \mathcal{X}_{f_J} \neq \emptyset$ and thus that $\text{rk } \Gamma_* \in \mathcal{I}$. \square

Lemma 5.7. *With notations from 1.2 we have that $D(\mathcal{E}) = D(\mathcal{G}_{\mathcal{E}})$ and $Q(\mathcal{E}) = Q(\mathcal{G}_{\mathcal{E}})$. In particular $D(\mathcal{G}_*: \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X}, \mathcal{C}})) = D(\mathcal{C} \rightarrow \text{Vect}(\mathcal{X}))$.*

Proof. We have to show that $\text{ranks}(\mathcal{E}) = \text{ranks}(\mathcal{G}_{\mathcal{E}})$. Since $\mathcal{P}_{\mathcal{C}}^* \mathcal{G}_{\mathcal{E}} \simeq \mathcal{E}$ we have $\text{ranks}(\mathcal{E}) \subseteq \text{ranks}(\mathcal{G}_{\mathcal{E}})$. For the converse, let $n \in \text{ranks}(\mathcal{G}_{\mathcal{E}})$ and $\xi: \text{Spec } L \rightarrow \text{Fib}_{\mathcal{X}, \mathcal{C}}$ be a point such that $\xi^* \mathcal{G}_{\mathcal{E}}$ has rank n . Let $\Gamma \in \text{Fib}_{\mathcal{X}, \mathcal{C}}(L)$ be the monoidal functor corresponding to ξ , so that $\xi^* \mathcal{G}_{\mathcal{E}} = \Gamma_{\mathcal{E}}$, and set $f = \text{rk } \Gamma: \mathcal{C} \rightarrow \mathbb{N}$, so that $f(\mathcal{E}) = n$. Since $\Gamma \in \text{Fib}_{\mathcal{X}, \mathcal{C}}^f \neq \emptyset$, by 5.6 we have $f \in \mathcal{I}$. By definition of f there exists $t: \text{Spec } \Omega \rightarrow \mathcal{X}$ such that $f(\mathcal{G}) = \text{rk } t^* \mathcal{G}$ for all $\mathcal{G} \in \mathcal{C}$. In particular $f(\mathcal{E}) = n = \text{rk } t^* \mathcal{E}$. By definition it follows that $n \in \text{ranks}(\mathcal{E})$. \square

We set $D(\mathcal{C}) = D(\mathcal{G}_*: \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X}, \mathcal{C}})) = D(\mathcal{C} \rightarrow \text{Vect}(\mathcal{X}))$. Since \mathcal{X} is quasi-compact, by 1.10 we have that $D(\mathcal{C}) = \text{Spec } S$ is affine. By abuse of notation we denote by $Q(\mathcal{E}) = Q(\mathcal{G}_{\mathcal{E}})$ the pullback along $D(\mathcal{C}) \rightarrow D(\mathcal{E}) = D(\mathcal{G}_{\mathcal{E}})$ of the vector bundle denoted by the same symbol. By the above lemma there is a map $\text{Fib}_{\mathcal{X}, \mathcal{C}} \rightarrow D(\mathcal{C})$. We define the (pseudo)-functor $\Omega: \text{Aff}/S \rightarrow$ (categories) by

$$\Omega(A) = \{R\text{-linear and strong monoidal functors } \Gamma: \mathcal{C} \rightarrow \text{Vect}(A) \text{ with fixed } \Gamma_{\mathcal{E}} \simeq Q(\mathcal{E}) \otimes_S S\}$$

Lemma 5.8. *The functor Ω is an affine scheme and $\text{Fr}(\mathcal{G}_*: \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X}, \mathcal{C}}))$ is the locus in Ω of functors Γ which are right exact, that is $\Gamma \in \text{Fib}_{\mathcal{X}, \mathcal{C}}$.*

Proof. By definition $\text{Fr}(\mathcal{G}_*: \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X}, \mathcal{C}}))(A)$ is the groupoid of $\xi \in \text{Fib}_{\mathcal{X}, \mathcal{C}}(A)$ together with isomorphisms $\xi^* \mathcal{G}_{\mathcal{E}} \simeq Q(\mathcal{E}) \otimes_S A$. Since objects ξ correspond to R -linear and strong monoidal functors $\Gamma: \mathcal{C} \rightarrow \text{Vect}(A)$ which are right exact and $\xi^* \mathcal{G}_{\mathcal{E}} \simeq \Gamma_{\mathcal{E}}$, the second part of the statement is clear.

For the first one, notice that, since the $Q(\mathcal{E})$ are vector bundles on $D(\mathcal{C})$, the sheaves

$$\underline{\text{Hom}}_{D(\mathcal{C})}(Q(\mathcal{E}), Q(\mathcal{E}')), \underline{\text{Iso}}_{D(\mathcal{C})}(Q(\mathcal{E}) \otimes Q(\mathcal{E}'), Q(\mathcal{E} \otimes \mathcal{E}')), \underline{\text{Iso}}(\mathcal{O}_S, Q(\mathcal{O}_{\mathcal{X}}))$$

are all affine schemes for $\mathcal{E}, \mathcal{E}' \in \mathcal{C}$. In particular

$$\overline{\Omega} = \left(\prod_{\mathcal{E} \rightarrow \mathcal{E}'} \underline{\text{Hom}}_{D(\mathcal{C})}(Q(\mathcal{E}), Q(\mathcal{E}')) \right) \times \left(\prod_{\mathcal{E}, \mathcal{E}'} \underline{\text{Iso}}_{D(\mathcal{C})}(Q(\mathcal{E}) \otimes Q(\mathcal{E}'), Q(\mathcal{E} \otimes \mathcal{E}')) \right) \times (\underline{\text{Iso}}(\mathcal{O}_S, Q(\mathcal{O}_{\mathcal{X}})))$$

where the product is taken over $D(\mathcal{C})$, is a very big affine scheme and Ω is a subfunctor of $\overline{\Omega} = \text{Spec } T$.

Set $Q(\mathcal{E}) \otimes_S T = Q(\mathcal{E})_T$. By construction there are maps $Q(\mathcal{E})_T \rightarrow Q(\mathcal{E}')_T$ for any map $\mathcal{E} \rightarrow \mathcal{E}'$ in \mathcal{C} , isomorphisms $Q(\mathcal{E})_T \otimes Q(\mathcal{E}')_T \rightarrow Q(\mathcal{E} \otimes \mathcal{E}')_T$ for all $\mathcal{E}, \mathcal{E}' \in \mathcal{C}$ and an isomorphism $\mathcal{O}_{\overline{\Omega}} \rightarrow Q(\mathcal{O}_{\mathcal{X}})_T$. We have that Ω is the locus in $\overline{\Omega}$ where the association $\mathcal{E} \mapsto Q(\mathcal{E})_T$ defines an R -linear strong monoidal functor. All these condition can be expressed by the vanishing of a map of vector bundles, which are closed relations. For example the locus where the association $Q(-)_T$ is symmetric is the locus where the difference between $Q(\mathcal{E})_T \otimes Q(\mathcal{E}')_T \rightarrow Q(\mathcal{E} \otimes \mathcal{E}')_T \rightarrow Q(\mathcal{E}' \otimes \mathcal{E})_T$ and $Q(\mathcal{E})_T \otimes Q(\mathcal{E}')_T \rightarrow Q(\mathcal{E}')_T \otimes Q(\mathcal{E})_T \rightarrow Q(\mathcal{E}' \otimes \mathcal{E})_T$ vanishes.

In conclusion Ω is a closed subscheme of $\overline{\Omega}$ and it is therefore affine. \square

We now assume the notation from 5.1, 2). In particular we set $G = \prod_{\mathcal{E}} \text{GL}(Q(\mathcal{E})) \rightarrow D(\mathcal{C})$.

Lemma 5.9. *The functor $\text{Fib}_{\mathcal{X},\mathcal{C}} \rightarrow \text{BG}$ is pseudo-affine and, in particular, $\text{Fr}(\mathcal{G}_*: \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}}))$ is pseudo-affine. If moreover the hypothesis of 5.1, 5) are met then the previous map and space are quasi-affine.*

Proof. By 2.5 and 2.6 it is enough to prove that $U = \text{Fr}(\mathcal{G}_*: \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}}))$ is pseudo-affine. We are going to use 5.8. Set $\Omega = \text{Spec } B$ and denote by $\Gamma: \mathcal{C} \rightarrow \text{Vect}(B)$ the tautological R -linear and strong monoidal functor. Given a finite test sequence \mathcal{T} in \mathcal{C} the sequence of maps $\Gamma_{\mathcal{T}}$ is a complex of free B -modules and denote by $\Omega_{\mathcal{T}}$ the locus in Ω where this complex is exact. By 4.1 the locus $\Omega_{\mathcal{T}}$ is a quasi-compact open subset of Ω . Moreover U is the intersection of all the $\Omega_{\mathcal{T}}$ for all finite test sequences \mathcal{T} . The fact that U is pseudo-affine follows from 4.2. In the hypothesis of 5.1, 5) only finitely many test sequences are needed and therefore U is a finite intersection of the $\Omega_{\mathcal{T}}$, hence it is quasi-affine. \square

Set $\mathcal{D} = \{\mathcal{G}_{\mathcal{E}}\}_{\mathcal{E} \in \mathcal{C}} \subseteq \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}})$ and let \mathcal{C}' be the subcategory of $\text{Vect } \text{BG}$ obtained by taking tensor products of the sheaves considered in 1.16 with respect to the map $\mathcal{C} \rightarrow \text{Vect}(D(\mathcal{C}))$, $\mathcal{E} \mapsto Q(\mathcal{E})$. By 1.16 the category \mathcal{C}' generates $\text{QCoh}(\text{BG})$. Set also $\omega: \text{Fib}_{\mathcal{X},\mathcal{C}} \rightarrow \text{BG}$.

Lemma 5.10. *We have $\omega^*\mathcal{C}' \subseteq \mathcal{D}$ and, in particular, \mathcal{D} generates $\text{QCoh}(\text{Fib}_{\mathcal{X},\mathcal{C}})$.*

Proof. The second statement follows from the first using 2.4 and the fact that ω is pseudo-affine. For the first, denote by $\mathcal{F}_{\mathcal{E}}$ the tautological sheaves defined over BG as in 1.16. By construction $\omega^*\mathcal{F}_{\mathcal{E}} \simeq \mathcal{G}_{\mathcal{E}}$. Using 5.5 we have $\mathcal{G}_{\mathcal{E}^\vee} \simeq (\mathcal{G}_{\mathcal{E}})^\vee \simeq g^*(\mathcal{F}_{\mathcal{E}})^\vee$ and, if $\text{Sym}^n \mathcal{E} \in \mathcal{C}$, $\mathcal{G}_{\text{Sym}^n \mathcal{E}} \simeq \text{Sym}^n \mathcal{G}_{\mathcal{E}} \simeq g^*(\text{Sym}^n \mathcal{F}_{\mathcal{E}})$ over $\text{Fib}_{\mathcal{X},\mathcal{C}}$, which concludes the proof. \square

We now deal with the case of $\text{Fib}_{\mathcal{X},\mathcal{C}}^f$ for $f \in \mathcal{I}$. The map $\text{Spec } R \rightarrow D(\mathcal{C})$ defined in 5.1, 4) is a flat closed immersion thanks to 1.11. Moreover it is easy to check that $G \times_{D(\mathcal{C})} R = \text{GL}_f$ and therefore $(\text{B}_{D(\mathcal{C})} G) \times_{D(\mathcal{C})} R = \text{BGL}_f$. Moreover by construction $\text{Fib}_{\mathcal{X},\mathcal{C}}^f$ is the locus of $\text{Fib}_{\mathcal{X},\mathcal{C}}$ where all the $\mathcal{G}_{\mathcal{E}}$ have rank $f(\mathcal{E})$. Hence $\text{Fib}_{\mathcal{X},\mathcal{C}}^f = \text{Fib}_{\mathcal{X},\mathcal{C}} \times_{D(\mathcal{C})} R$. Taking into account 2.4 we obtain:

Proposition 5.11. *Claims in Theorem 5.1, 4) hold.*

It remains to show Theorem 5.1, 5). So we assume its hypothesis. We have already shown that $U = \text{Fr}(\mathcal{G}_*: \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X},\mathcal{C}}))$ is quasi-affine in 5.9. For $T \subseteq \mathcal{C}$ set $G_T = \prod_{\mathcal{E} \in T} \text{GL}(Q(\mathcal{E})) \rightarrow D(\mathcal{C})$. In particular $G = G_{\mathcal{C}} = G_J \times G_{\mathcal{C}-J}$ and $\text{BG} = \text{BG}_J \times \text{BG}_{\mathcal{C}-J}$. Consider the Cartesian diagram

$$\begin{array}{ccccc} U & \longrightarrow & D(\mathcal{C}) & & \\ \downarrow & & \downarrow & & \\ U' & \longrightarrow & \text{BG}_{\mathcal{C}-J} & \longrightarrow & D(\mathcal{C}) \\ \downarrow \alpha & & \downarrow & & \downarrow \\ \text{Fib}_{\mathcal{X},\mathcal{C}} & \longrightarrow & \text{BG} & \longrightarrow & \text{BG}_J \end{array}$$

Lemma 5.12. *The map $U \rightarrow U'$ has a section and, in particular, U' is a closed subscheme of U , hence quasi-affine.*

Proof. We need to show that the torsor $U' \rightarrow \text{BG}_{\mathcal{C}-J}$ is trivial. For this, it is enough to show that $U' \rightarrow \text{BG}$ is trivial. This last map is given by the collection $(\alpha^*\mathcal{G}_{\mathcal{E}})_{\mathcal{E} \in \mathcal{C}}$ and therefore we must show that all those sheaves are free.

Let $\mathcal{S} \subseteq \mathcal{C}$ be the subset of \mathcal{E} such that $\alpha^*\mathcal{G}_{\mathcal{E}}$ is free. By construction $J \subseteq \mathcal{S}$. On the other hand in 5.5 we have seen that if $\mathcal{E} \in \mathcal{C}$ is such that $t(\mathcal{E}) \in \mathcal{C}$ then $\mathcal{G}_{t(\mathcal{E})} \simeq t(\mathcal{G}_{\mathcal{E}})$ for any of the following operations of sheaves $t = -^\vee, \text{Sym}^m, \Lambda^m$. In particular $\alpha^*\mathcal{G}_{t(\mathcal{E})} \simeq t(\alpha^*\mathcal{G}_{\mathcal{E}})$ and

therefore $\mathcal{E} \in \mathcal{S}$ implies $t(\mathcal{E}) \in \mathcal{S}$. By a direct check it is also clear that $\mathcal{G}_{\mathcal{E} \otimes \mathcal{E}'} \simeq \mathcal{G}_{\mathcal{E}} \otimes \mathcal{G}_{\mathcal{E}'}$ for $\mathcal{E}, \mathcal{E}' \in \mathcal{C}$ and, if $\mathcal{E} \oplus \mathcal{E}' \in \mathcal{C}$, that $\mathcal{G}_{\mathcal{E} \oplus \mathcal{E}'} \simeq \mathcal{G}_{\mathcal{E}} \oplus \mathcal{G}_{\mathcal{E}'}$. Thus \mathcal{S} is closed under tensor product and, when it makes sense, under direct sum.

Using the first condition on \mathcal{C} we can conclude that $\mathcal{S} = \mathcal{C}$. Since $U \rightarrow U'$ is a $G_{\mathcal{C}-J}$ -torsor and therefore affine, the existence of a section tell us that U' is a closed subscheme of U and therefore is quasi-affine. \square

As a consequence of the previous result we have that $\text{Fib}_{\mathcal{X}, \mathcal{C}} \simeq [U'/G_J]$ with U' quasi-affine. We want to show that $\text{Fib}_{\mathcal{X}, \mathcal{C}} \simeq [V/\text{GL}_n]$ for some quasi-affine scheme V . Since J is finite, there is a finite decomposition of $D(\mathcal{C}) = \sqcup_i D_i$ into open and closed subsets such that $Q(\mathcal{E})|_{U_i}$ has constant rank for $\mathcal{E} \in J$. In particular we have that $(G_J)|_{U_i}$ is a finite product of GL_m 's. The following lemma concludes the proof of Theorem 5.1, 5) and therefore of Theorem 5.1 itself.

Lemma 5.13. *Let $m_1, \dots, m_h \in \mathbb{N}$ and $n > m_1 + \dots + m_h$. Then there is an injective map*

$$G = \text{GL}_{m_1} \times \dots \times \text{GL}_{m_h} \rightarrow \text{GL}_n$$

over \mathbb{Z} such that $\text{B}G \rightarrow \text{BGL}_n$ is quasi-affine. In particular for any quasi-affine scheme U with an action of G one can write

$$[U/G] \simeq [V/\text{GL}_n]$$

for some quasi-affine scheme V with an action of GL_n .

Proof. The last claim follows from the first because $[U/G] \rightarrow \text{B}G \rightarrow \text{BGL}_n$ would be quasi-affine because composition of quasi-affine maps.

Consider as $G \rightarrow \text{GL}_n$ the map sending matrices M_1, \dots, M_h to the $n \times n$ matrix having the M_j on the diagonal and 1 in the remaining spots of the diagonal. The corresponding map $\text{B}G \rightarrow \text{BGL}_n$, in terms of vector bundles, is induced by $\mathcal{H} = (\bigoplus_i \mathcal{F}_i) \oplus \mathcal{O}^{n-m}$, where $m = \sum_i m_i$ and \mathcal{F}_i is the rank m_i tautological vector bundle given by $\text{B}G \rightarrow \text{B}G_i$. Denote by $h: W \rightarrow \text{B}G$ the corresponding GL_n -torsor. We have to show that W is quasi-affine and, since W is an algebraic space, it is enough to show that it is pseudo-affine thanks to 2.8. It is enough to show that the sheaves over $\text{B}G$ coming from BGL_n , whose category we denote by \mathcal{D}_n , generates $\text{QCoh}(\text{B}G)$: $h^* \mathcal{D}_n$ are made of free sheaves and, since h is affine, it generates $\text{QCoh}(W)$ thanks to 2.4.

We have $\mathcal{H} = \mathcal{F}_i \oplus \mathcal{Q}_i$. In particular there is a surjective map $\mathcal{H} \rightarrow \mathcal{F}_i$. Moreover, since $\text{Sym}^q(\mathcal{H}) = (\text{Sym}^q \mathcal{F}_i) \oplus \mathcal{K}_i$, there is also a surjective map $\mathcal{D}_n \ni (\text{Sym}^m \mathcal{H})^\vee \rightarrow (\text{Sym}^m \mathcal{F}_i)^\vee$. It follows that the sheaves $\mathcal{F}_i, (\text{Sym}^q \mathcal{F}_i)^\vee$ for $q \in \mathbb{N}$ are generated by \mathcal{D}_n , which therefore generates $\text{QCoh}(\text{B}G)$ thanks to 1.16. \square

5.2. Comparing vector bundles. In this section we want to discuss some relations between the vector bundles on \mathcal{X} and on $\text{Fib}_{\mathcal{X}, \mathcal{C}}$.

Proposition 5.14. *Let \mathcal{X} and \mathcal{C} be as in 5.1. The following conditions are equivalent:*

- 1) *(if \mathcal{X} is quasi-separated (see [Ton20, Def 1.4])) the morphism $\mathcal{O}_{\text{Fib}_{\mathcal{X}, \mathcal{C}}} \rightarrow \mathcal{P}_{\mathcal{C}*} \mathcal{O}_{\mathcal{X}}$ is an isomorphism;*
- 2) *the functor $\text{Vect}(\text{Fib}_{\mathcal{X}, \mathcal{C}}) \rightarrow \text{Vect}(\mathcal{X})$ is fully faithful;*
- 3) *the functor $\mathcal{G}_*: \mathcal{C} \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X}, \mathcal{C}})$ is fully faithful.*
- 4) *the functor $\mathcal{G}_*(\mathcal{C}) \rightarrow \text{Vect}(\mathcal{X})$ is faithful*

Proof. 1. \iff 2. We assume \mathcal{X} quasi-compact and quasi-separated. Since $\text{Fib}_{\mathcal{X}, \mathcal{C}}$ is quasi-compact and quasi-separated by 5.1, the functor $\mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \text{Fib}_{\mathcal{X}, \mathcal{C}}$ is quasi-compact and quasi-separated as well, so that pushing forward quasi-coherent sheaves works as expected (see [Ton20, Prop 1.15]). In particular $\mathcal{P}_{\mathcal{C}*} \mathcal{O}_{\mathcal{X}}$ is a quasi-coherent sheaf.

For $\mathcal{H} \in \mathrm{QCoh}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}})$ consider the morphism

$$\delta_{\mathcal{H}}: \mathrm{Hom}(\mathcal{H}, \mathcal{O}_{\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}}) \rightarrow \mathrm{Hom}(\mathcal{H}|_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$$

This map can also be obtained applying $\mathrm{Hom}(\mathcal{H}, -)$ on the map $\mathcal{O}_{\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}} \rightarrow \mathcal{P}_{\mathcal{C}*} \mathcal{O}_{\mathcal{X}}$:

$$\mathrm{Hom}(\mathcal{H}, \mathcal{O}_{\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}}) \rightarrow \mathrm{Hom}(\mathcal{H}, \mathcal{P}_{\mathcal{C}*} \mathcal{O}_{\mathcal{X}}) \simeq \mathrm{Hom}(\mathcal{H}|_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$$

In particular Yoneda's lemma tells us that 1. is equivalent to the fact that $\delta_{\mathcal{H}}$ is an isomorphism for all $\mathcal{H} \in \mathrm{QCoh}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}})$. On the other hand, condition 2. is equivalent to the fact that all maps $\delta_{\mathcal{H}}$ are isomorphisms for $\mathcal{H} \in \mathrm{Vect}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}})$. Thus clearly 1. \implies 2.. The converse instead follows from 1.6: $\mathrm{QCoh}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}) \rightarrow \mathrm{L}_R(\mathrm{Vect}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}), R)$ is fully faithful because $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ has the resolution property by 5.1.

2. \implies 3. \iff 4. It is enough to recall that $\mathcal{C} \xrightarrow{\mathcal{G}_*} \mathrm{Vect}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}) \rightarrow \mathrm{Vect}(\mathcal{X})$ is the inclusion.
4. \implies 2. We have to show that the map

$$\mathrm{Hom}(\mathcal{H}, \mathcal{H}') \rightarrow \mathrm{Hom}(\mathcal{H}|_{\mathcal{X}}, \mathcal{H}'|_{\mathcal{X}'})$$

is an isomorphism for all $\mathcal{H}, \mathcal{H}' \in \mathrm{Vect}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}})$. Since $\mathcal{G}_*(\mathcal{C})$ generates $\mathrm{QCoh}(\mathcal{X})$ by 5.1 and \mathcal{H} is finitely presented, there exists an exact sequence

$$\mathcal{G}_{\mathcal{E}'} \rightarrow \mathcal{G}_{\mathcal{E}} \rightarrow \mathcal{H} \rightarrow 0$$

for some $\mathcal{E}, \mathcal{E}' \in \mathcal{C}^{\oplus}$. As $\mathrm{Hom}(-, \mathcal{H}')$ and $\mathrm{Hom}(-|_{\mathcal{X}}, \mathcal{H}'|_{\mathcal{X}'})$ are both exact on the above sequence, we can assume $\mathcal{H} = \mathcal{G}_{\bar{\mathcal{E}}}$ for some $\bar{\mathcal{E}} \in \mathcal{C}^{\oplus}$.

Considering a resolution for \mathcal{H}'^{\vee} , dualizing and using that $\mathcal{G}_{\bar{\mathcal{E}}}^{\vee} \simeq \mathcal{G}_{\mathcal{E}^{\vee}}$ by 5.5, we also obtain an exact sequence

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{G}_{\mathcal{E}} \rightarrow \mathcal{G}_{\mathcal{E}'}$$

for some $\mathcal{E}, \mathcal{E}' \in \mathcal{C}^{\oplus}$. Moreover also this sequence restricts to an exact sequence on \mathcal{X} because \mathcal{H}' is locally free: $\mathrm{Coker}(\mathcal{H} \rightarrow \mathcal{G}_{\mathcal{E}})$ is locally free. In particular $\mathrm{Hom}(\mathcal{G}_{\bar{\mathcal{E}}}, -)$ and $\mathrm{Hom}(\bar{\mathcal{E}}, -|_{\mathcal{X}'})$ are exact on the above sequence and therefore we can also assume $\mathcal{H}' = \mathcal{G}_{\mathcal{E}}$ for some $\mathcal{E} \in \mathcal{C}^{\oplus}$.

Splitting $\bar{\mathcal{E}}$ and \mathcal{E} into direct sums of sheaves in \mathcal{C} we are left to prove that

$$\mathrm{Hom}(\mathcal{G}_{\bar{\mathcal{E}}}, \mathcal{G}_{\mathcal{E}}) \rightarrow \mathrm{Hom}(\bar{\mathcal{E}}, \mathcal{E})$$

is an isomorphism for $\bar{\mathcal{E}}, \mathcal{E} \in \mathcal{C}$. By hypothesis this map is injective. On the other hand it is surjective because $\mathcal{C} \xrightarrow{\mathcal{G}_*} \mathrm{Vect}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}) \rightarrow \mathrm{Vect}(\mathcal{X})$ is the inclusion. \square

Conjecture 5.15. *Let \mathcal{X} and \mathcal{C} be as in 5.1. Then the equivalent conditions of Proposition 5.14 holds.*

Remark 5.16. Conjecture 5.15 holds in two special cases. The first one is when \mathcal{X} is quasi-compact with quasi-affine diagonal and \mathcal{C} generates $\mathrm{QCoh}(\mathcal{X})$: by 3.2 the map $\mathcal{X} \rightarrow \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ is actually an equivalence. The second is when \mathcal{X} is quasi-compact and quasi-separated and $\mathcal{C} = \{\mathcal{O}_{\mathcal{X}}\}$, the simplest category: the Conjecture is verified thanks to 5.20, that we are going to prove.

Proposition 5.17. *Let \mathcal{X} be a quasi-compact fibered category over R and \mathcal{Y} a quasi-compact fpqc stack with quasi-affine diagonal and with the resolution property. Then any map $f: \mathcal{X} \rightarrow \mathcal{Y}$ has a canonical factorization $\mathcal{X} \rightarrow \mathrm{Fib}_{\mathcal{X}, \mathrm{Vect}(\mathcal{X})} \rightarrow \mathcal{Y}$.*

Proof. The canonical extension $\mathrm{Fib}_{\mathcal{X}, \mathrm{Vect}(\mathcal{X})} \rightarrow \mathcal{Y}$ is the composition of $\mathrm{Fib}_{\mathcal{X}, \mathrm{Vect}(\mathcal{X})} \rightarrow \mathrm{Fib}_{\mathcal{Y}, \mathrm{Vect}(\mathcal{Y})}$, $\Gamma \mapsto \Gamma \circ f^*$ and the equivalence $\mathcal{Y} \rightarrow \mathrm{Fib}_{\mathcal{Y}, \mathrm{Vect}(\mathcal{Y})}$ (see 3.2). This can also be described, again using 3.2, as the map corresponding to the composition

$$\mathrm{Vect}(\mathcal{Y}) \rightarrow \mathrm{Vect}(\mathcal{X}) \xrightarrow{\mathcal{G}_*} \mathrm{Vect}(\mathrm{Fib}_{\mathcal{X}, \mathrm{Vect}(\mathcal{X})})$$

\square

Conjecture 5.18. *Let \mathcal{X} be a quasi-compact fibered category over R . The morphism $\mathcal{X} \rightarrow \text{Fib}_{\mathcal{X}, \mathcal{C}}$ is universal among maps from \mathcal{X} to quasi-compact fpqc stacks with quasi-affine diagonal and with the resolution property, that is*

$$(5.1) \quad \text{Hom}(\text{Fib}_{\mathcal{X}, \text{Vect}(\mathcal{X})}, \mathcal{Y}) \rightarrow \text{Hom}(\mathcal{X}, \mathcal{Y})$$

is an equivalence for all such stacks \mathcal{Y} .

Proposition below shows that Conjecture 5.15 for $\mathcal{C} = \text{Vect}(\mathcal{X})$ is equivalent to Conjecture 5.18.

Proposition 5.19. *Conjecture 5.18 holds if and only if*

$$\text{Vect}(\text{Fib}_{\mathcal{X}, \text{Vect}(\mathcal{X})}) \rightarrow \text{Vect}(\mathcal{X})$$

is faithful. In this case the above map is an equivalence.

Proof. As the composition $\text{Vect}(\mathcal{X}) \xrightarrow{\mathcal{G}_*} \text{Vect}(\text{Fib}_{\mathcal{X}, \text{Vect}(\mathcal{X})}) \rightarrow \text{Vect}(\mathcal{X})$ is the identity, we see that if $\text{Vect}(\text{Fib}_{\mathcal{X}, \text{Vect}(\mathcal{X})}) \rightarrow \text{Vect}(\mathcal{X})$ is fully faithful then it is an equivalence. Moreover by 5.14 faithfulness also implies the fullness. Assume this is true. If we start by a functor $\text{Fib}_{\mathcal{X}, \text{Vect}(\mathcal{X})} \rightarrow \mathcal{Y}$, which is completely determined by the pullback

$$\text{Vect}(\mathcal{Y}) \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X}, \text{Vect}(\mathcal{X})})$$

by 3.2, the new map $\text{Fib}_{\mathcal{X}, \text{Vect}(\mathcal{X})} \rightarrow \mathcal{Y}$ obtained corresponds to

$$\text{Vect}(\mathcal{Y}) \rightarrow \text{Vect}(\text{Fib}_{\mathcal{X}, \text{Vect}(\mathcal{X})}) \rightarrow \text{Vect}(\mathcal{X}) \xrightarrow{\mathcal{G}_*} \text{Vect}(\text{Fib}_{\mathcal{X}, \text{Vect}(\mathcal{X})})$$

and therefore is canonically isomorphic to the original one.

Conversely assume $\mathcal{X} \rightarrow \text{Fib}_{\mathcal{X}, \text{Vect}(\mathcal{X})}$ is universal. Given $h \in \mathbb{N}$ we define \mathcal{Y}_h as the stack of vector bundles whose local rank is bounded above by h . In other words

$$\mathcal{Y}_h = \bigsqcup_{l=0}^h \text{BGl}_l$$

which is a quasi-compact stack with affine diagonal and the resolution property. Assuming that (5.1) is an equivalence for all \mathcal{Y}_h means that

$$\text{Vect}(\text{Fib}_{\mathcal{X}, \text{Vect}(\mathcal{X})}) \rightarrow \text{Vect}(\mathcal{X})$$

induces an equivalence on the corresponding groupoids. We have to show that, if $\mathcal{E}, \mathcal{H} \in \text{Vect}(\text{Fib}_{\mathcal{X}, \text{Vect}(\mathcal{X})})$ and $\phi: \mathcal{E} \rightarrow \mathcal{H}$ is a morphism and $\phi|_{\mathcal{X}} = 0$ then $\phi = 0$. Consider the isomorphism

$$\psi: (\mathcal{E} \oplus \mathcal{H}) \rightarrow (\mathcal{E} \oplus \mathcal{H}), \quad e \oplus h \mapsto e \oplus (h + \phi(e))$$

By construction $\psi|_{\mathcal{X}} = \text{id}$. Thus $\psi = \text{id}$ and $\phi = 0$ as required. \square

5.3. The case $\mathcal{C} = \{\mathcal{O}_{\mathcal{X}}\}$. In this section we consider the simple case $\mathcal{C} = \{\mathcal{O}_{\mathcal{X}}\}$. Notice that $\text{Sym}^n \mathcal{O}_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{X}}$ so the hypothesis of 5.1 are satisfied.

Proposition 5.20. *Let \mathcal{X} be a quasi-compact fibered category over R . Then $\text{Fib}_{\mathcal{X}, \{\mathcal{O}_{\mathcal{X}}\}}$ is the intersection of all quasi-compact open subsets of $\text{Spec } H^0(\mathcal{O}_{\mathcal{X}})$ containing the image of $\mathcal{X} \rightarrow \text{Spec } H^0(\mathcal{O}_{\mathcal{X}})$.*

If \mathcal{X} is quasi-compact and quasi-separated then $\text{Vect}(\text{Fib}_{\mathcal{X}, \{\mathcal{O}_{\mathcal{X}}\}}) \rightarrow \text{Vect}(\mathcal{X})$ is fully faithful, or, equivalently, $\mathcal{O}_{\text{Fib}_{\mathcal{X}, \{\mathcal{O}_{\mathcal{X}}\}}} \rightarrow \mathcal{P}_{\{\mathcal{O}_{\mathcal{X}}\}} \mathcal{O}_{\mathcal{X}}$ is an isomorphism. Moreover $\mathcal{X} \rightarrow \text{Fib}_{\mathcal{X}, \{\mathcal{O}_{\mathcal{X}}\}}$ is universal among maps from \mathcal{X} to pseudo-affine sheaves.*

Proof. We can assume $R = H^0(\mathcal{O}_{\mathcal{X}})$. Set $\mathcal{C} = \{\mathcal{O}_{\mathcal{X}}\}$. If A is an R -algebra, there is a unique R -linear and strong monoidal functor $\Gamma^A: \mathcal{C} \rightarrow \text{Vect}(A)$, namely $\Gamma^A = H^0(-) \otimes_R A$. This means that $\text{Fib}_{\mathcal{X}, \mathcal{C}}$ is a subfunctor of $\text{Spec } R$. We have that $\Gamma^A \in \text{Fib}_{\mathcal{X}, \mathcal{C}}(A)$ if and only if, given a test sequence \mathcal{T}_* for \mathcal{C} , Γ^A is exact on it. As

$$\Gamma^A(\mathcal{T}_*) = H^0(\mathcal{T}_*) \otimes_R A$$

we can conclude that $\text{Fib}_{\mathcal{X}, \mathcal{C}}$ is the locus where all complex of free R -modules $H^0(\mathcal{T}_*)$ are exact. By 4.1, this is an intersection of quasi-compact open subsets of $\text{Spec } R$ containing the image of $\mathcal{X} \rightarrow \text{Spec } R$. Conversely if $\mathcal{X} \rightarrow U \subseteq \text{Spec } R$ and U is a quasi-compact open subset, say $U = \text{Spec } R - V(a_1, \dots, a_n)$, then U is the locus where $(a_1, \dots, a_n): R^n \rightarrow R$ is surjective and, in particular,

$$\mathcal{O}_{\mathcal{X}}^n \xrightarrow{(a_1, \dots, a_n)} \mathcal{O}_{\mathcal{X}} \rightarrow 0$$

is a test sequence. Therefore $\text{Fib}_{\mathcal{X}, \mathcal{C}} \subseteq U$ as required.

Now assume that \mathcal{X} (and therefore $\mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \text{Fib}_{\mathcal{X}, \mathcal{C}}$) is quasi-compact and quasi-separated. Set $\pi: \mathcal{X} \rightarrow \text{Spec } R$. By construction $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\text{Spec } R}$. From the Cartesian diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\text{id}} & \mathcal{X} \\ \downarrow \mathcal{P}_{\mathcal{C}} & & \downarrow \pi \\ \text{Fib}_{\mathcal{X}, \mathcal{C}} & \xrightarrow{u} & \text{Spec } R \end{array}$$

the flatness of u and [Ton20, Prop 1.15] we can conclude that

$$\mathcal{O}_{\text{Fib}_{\mathcal{X}, \mathcal{C}}} \simeq u^* \mathcal{O}_{\text{Spec } R} \simeq u^*(\pi_* \mathcal{O}_{\mathcal{X}}) \simeq \mathcal{P}_{\mathcal{C}*} \mathcal{O}_{\mathcal{X}}$$

By 5.14 this is equivalent to the fully faithfulness of $\text{Vect}(\text{Fib}_{\mathcal{X}, \mathcal{C}}) \rightarrow \text{Vect}(\mathcal{X})$.

For the last claim, if $\mathcal{X} \rightarrow U$ is a map to a quasi-affine scheme consider the diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\quad} & \text{Fib}_{\mathcal{X}, \mathcal{C}} & \dashrightarrow & V & \longrightarrow & \text{Spec } R \\ & \searrow & & & \downarrow & & \downarrow \\ & & & & U & \longrightarrow & \text{Spec } H^0(\mathcal{O}_U) \end{array}$$

where the square one is Cartesian. The dashed arrow exists thanks to the description of $\text{Fib}_{\mathcal{X}, \mathcal{C}}$ as intersection. From the same diagram we can see that $\text{Fib}_{\mathcal{X}, \mathcal{C}} \rightarrow U$ is uniquely determined. The general case of a pseudo-affine sheaf follows easily from 4.2. \square

Proposition 5.21. *Let \mathcal{X} be a quasi-compact fibered category. If $\mathcal{X} \rightarrow \text{Spec } H^0(\mathcal{O}_{\mathcal{X}})$ has open image then $\text{Fib}_{\mathcal{X}, \{\mathcal{O}_{\mathcal{X}}\}}$ coincides with this image. Conversely if $\text{Fib}_{\mathcal{X}, \{\mathcal{O}_{\mathcal{X}}\}}$ is an algebraic stack, $\mathcal{X} \rightarrow \text{Spec } R$ is a finitely presented algebraic stack and $H^0(\mathcal{O}_{\mathcal{X}})$ is a Noetherian Jacobson ring then $\mathcal{X} \rightarrow \text{Spec } H^0(\mathcal{O}_{\mathcal{X}})$ has open image.*

Proof. Set $\mathcal{C} = \{\mathcal{O}_{\mathcal{X}}\}$. The first claim follows from 5.20, so let us focus on the converse.

We can assume $R = H^0(\mathcal{O}_{\mathcal{X}})$. By 2.8 and 5.20, the fact that $\text{Fib}_{\mathcal{X}, \mathcal{C}}$ is algebraic means that it is quasi-affine. Moreover, since $H^0(\mathcal{O}_{\text{Fib}_{\mathcal{X}, \mathcal{C}}}) = R$, it follows that $V = \text{Fib}_{\mathcal{X}, \mathcal{C}}$ is open inside $\text{Spec } R$. By Chevalley's theorem [Aut19, Tag 054K] the image U of $\mathcal{X} \rightarrow \text{Spec } R$ is constructible. In conclusion, from 5.20, we can conclude that V is the smallest open of $\text{Spec } R$ containing U . Assume by contradiction $U \neq V$. As $V - U$ is constructible, we need to show that a constructible subset Q of $\text{Spec } R$ contains a closed subset. We can easily reduce to the case that Q is open. As R is a Jacobson ring, Q contains a closed point of $\text{Spec } R$ as required. \square

Example 5.22. We show an example of an integral scheme X of finite type over a field k such that

- the stack $\mathrm{Fib}_{X, \{\mathcal{O}_X\}}$ is not algebraic.
- the map $X \rightarrow \mathrm{Fib}_{X, \{\mathcal{O}_X\}}$ is not surjective, which shows that not all fiber functors (locally) comes from a section of X .

By 5.21, the idea is to look for an X such that the image of $X \rightarrow \mathrm{Spec} H^0(\mathcal{O}_X)$ is not open. Let $A = k[x, y, z]$ and $B = k[x/y, y, z/y]$ and notice that

$$A \subseteq B \subseteq A_y \subseteq A_{xy} = B_x$$

We define X as the scheme obtained by gluing $\mathrm{Spec} A_x$ and $\mathrm{Spec} B$ along $\mathrm{Spec} A_{xy}$. By construction there are maps

$$\mathrm{Spec} A - V(x, y) = \mathrm{Spec} A_x \cup \mathrm{Spec} A_y \rightarrow X \xrightarrow{f} \mathrm{Spec} A$$

As the first map is dominant because $\mathrm{Spec} A_x \subseteq X$ is open, we obtain that $H^0(\mathcal{O}_X) = A$. Moreover

$$f^{-1}(V(x, y)) = \mathrm{Spec}(B/(x, y)) = \mathrm{Spec}(k[x/y, z/y]) \rightarrow \mathrm{Spec} A$$

is given by $A \rightarrow k[x/y, z/y]$, $x, y, z \rightarrow 0$. It follows that the image of $X \rightarrow \mathrm{Spec} A$ is $\mathrm{Im} = \mathrm{Spec} A - V(x, y) \sqcup \{(x, y, z)\}$.

By 5.21 the stack $\mathrm{Fib}_{X, \{\mathcal{O}_X\}}$ cannot be algebraic because Im is not open. Moreover we claim that $P = (x, y) \in \mathrm{Fib}_{X, \{\mathcal{O}_X\}} - \mathrm{Im}$. By 5.20 we have to show that if $\mathrm{Im} \subseteq U \subseteq \mathrm{Spec} A$ is open then $P \in U$. This is true because otherwise we would have the contradiction

$$P \in (\mathrm{Spec} A - U) \subseteq \mathrm{Spec} A - \mathrm{Im} = V(P) - \{(x, y, z)\} \subseteq V(P) = \overline{\{P\}}$$

REFERENCES

- [Aut19] The Stacks Project Authors, *The stacks project*.
- [BC14] Martin Brandenburg and Alexandru Chirvasitu, *Tensor functors between categories of quasi-coherent sheaves*, Journal of Algebra **399** (2014), 675–692.
- [Bha16] Bhargav Bhatt, *Algebraization and Tannaka duality*, Cambridge Journal of Mathematics **4** (2016), no. 4, 403–461.
- [Bra14] Martin Brandenburg, *Tensor categorical foundations of algebraic geometry*, Ph.D. thesis, 2014, p. 251.
- [Bro13] Michael Broshi, *G-torsors over a Dedekind scheme*, Journal of Pure and Applied Algebra **217** (2013), no. 1, 11–19.
- [Del90] Pierre Deligne, *Catégories tannakiennes*, The Grothendieck Festschrift, vol II, vol. 87, Birkhäuser, Boston, MA, 1990, pp. 111–195.
- [Gro60] Alexander Grothendieck, *EGAII - Étude globale élémentaire de quelques classes de morphismes - Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné)*, Inst. Hautes Études Sci. Publ. Math. 24, 1960.
- [Gro13] Philipp Gross, *Tensor generators on schemes and stacks*, Arxiv Article (2013), 21.
- [HR14] Jack Hall and David Rydh, *Coherent Tannaka duality and algebraicity of Hom-stacks*, arXiv:1405.7680 (2014), 32.
- [LMB05] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, first ed., Springer, 2005.
- [Lur04] Jacob Lurie, *Tannaka Duality for Geometric Stacks*, arXiv:math/0412266 (2004), 14.
- [Ray67] Michel Raynaud, *Un critère d'effectivité de descente*, Séminaire Samuel. Algèbre commutative **2** (1967), no. Talk no 5, 1–22.
- [Sav06] Valentin Savin, *Tannaka duality on quotient stacks*, Manuscripta Mathematica **119** (2006), no. 3, 287–303.
- [Sch12] Daniel Schäppi, *A characterization of categories of coherent sheaves of certain algebraic stacks*, arXiv:1206.2764 (2012), 64.
- [Sch13] ———, *The formal theory of tannaka Duality*, Asterisque **357** (2013), no. 357, 1–150.
- [Ton20] Fabio Tonini, *Sheafification of linear functors*, preprint (2020).
- [Tot02] Burt Totaro, *The resolution property for schemes and stacks*, Journal für die Reine und Angewandte Mathematik **577** (2002), 23.

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